

Abelian surfaces with two plane cubic curve fibrations and Calabi-Yau threefolds

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1. Introduction

Abelian surfaces of small degree are contained in nodal Calabi-Yau 3-folds, similarly many Calabi-Yau 3-folds of small degree specialize to nodal Calabi-Yau 3-folds with abelian surfaces on them. The first assertion is intimately connected with the fact that the moduli space of abelian surfaces of small degree is uniruled: An abelian surface on a Calabi-Yau 3-fold moves in a linear pencil, and therefore gives rise to a \mathbf{P}^1 in the moduli space of abelian surfaces. This idea was taken up and explored by Gross and Popescu [GP1, GP2] starting with a very singular Calabi-Yau variety, the secant variety of an elliptic normal curve. The translation scrolls inside the secant variety are degenerate abelian surfaces and form a \mathbf{P}^1 on the boundary of the moduli of abelian surfaces. They show that the secant variety deforms to nodal Calabi-Yau 3-folds with only isolated singularities and with a pencil of abelian surfaces as long as the degree of the elliptic curve is less than 11. This limit is related to the Del Pezzo bound for the possible smoothing of minimal elliptic surface singularities.

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We explore a similar setting. In a \mathbf{P}^2 -scroll over an elliptic curve, any anticanonical divisor, if there is one, is a, possibly degenerate, abelian surface. If one can glue two \mathbf{P}^2 -scrolls over elliptic curves along an anticanonical divisor, the union is a singular Calabi-Yau 3-fold. Furthermore, if the anticanonical divisor moves in a pencil on at least one of the two scrolls, then we are in a position like above.

We start in section 2 by asking for smooth abelian surfaces with two pencils of plane cubic curves on it. The two pencils would then define \mathbf{P}^2 -scrolls whose union is Calabi-Yau. It turns out that purely numerical considerations bound the degree of these abelian surfaces by 18. This bound is obtained by abelian surfaces which form the complete intersection $((0, 3), (3, 0))$ in the $\mathbf{P}^2 \times \mathbf{P}^2$ with its Segre embedding. For each even degree $10 \leq d \leq 18$ there are numerical possibilities which are realized. In this paper we study the associated elliptic scrolls and Calabi-Yau 3-folds in the case $d = 12$, i.e. the case of abelian surfaces embedded linearly normally in \mathbf{P}^5 .

In section 3 we find and describe the abelian surfaces of degree 12 and the two scrolls defined by their pencils of plane cubic curves. The union of the two scrolls is a non-normal Calabi-Yau 3-fold of degree 12.

In sections 4 and 5 a separate approach leads to constructions via projected Del Pezzo 3-folds of non-normal Calabi-Yau 3-folds in degrees 10, 11, 12 and 13. The projected Del Pezzo 3-folds are bilinked to the non-normal Calabi-Yau 3-folds. In the last three sections we prepare the argument that the reducible Calabi-Yau 3-folds described in section 3 may also be obtained via bilinkage from projected Del Pezzo 3-folds.

After finding equations for elliptic scrolls in section 6, we devote section 7 to the Heisenberg symmetry of elliptic scrolls and provide a description of the family of Calabi-Yau 3-folds that are unions of two scrolls.

More precisely, let H_6 be the Heisenberg group of level 6 and let N_6 be its normalizer in $GL(6, \mathbf{C})$. In N_6 there is a natural involution ι which restricts to the abelian surfaces as multiplication by -1 . Let $G_6 = \langle H_6, \iota \rangle$. Then the space of cubics in \mathbf{P}^5 contains a 4-dimensional vector space of G_6 -invariant pencils.

We let $\mathbf{P} = \mathbf{P}^3$ be the parameter space for these G_6 -invariant pencils of cubics. H_6 contains a subgroup isomorphic to H_2 and four subgroups $H(K_1), \dots, H(K_4)$ of index 3 containing this subgroup. For every subgroup $H(K_i)$ there is a set of three lines in \mathbf{P}^5 containing $H(K_i)$ in its stabilizer and left invariant by the action of H_6 . Similarly there exists for each subgroup $H(K_i)$ a line l_i in \mathbf{P} parametrizing pencils of cubics which contain the corresponding three lines.

A general point on any of these four lines l_i in \mathbf{P} corresponds to a pencil of cubics which defines an elliptic scroll singular along the corresponding three lines in \mathbf{P}^5 . The scroll is residual to three \mathbf{P}^3 's in the complete intersection of the two cubics.

Between each pair of lines l_i, l_j , there is a $1 : 1$ correspondence defined by the pairs of points which correspond to elliptic scrolls that intersect along a possibly degenerate abelian surface. The lines spanned by corresponding points form a conic section in the Grassmannian of lines in \mathbf{P} . Altogether there are 6 disjoint conic

sections in the Grassmannian of lines in \mathbf{P} which parametrize abelian surfaces with two plane cubic curve fibrations.

In the final section we state and prove the main theorem of the paper: Let X_E and X_F be elliptic scrolls in \mathbf{P}^5 that intersect precisely along a $(1, 6)$ -polarized abelian surface.

Theorem 8.3. *The reducible 3-fold $Y = X_E \cup X_F$ is a degeneration of irreducible non-normal Calabi-Yau 3-folds of degree 12 in \mathbf{P}^5 . The general such 3-fold is singular precisely along 6 disjoint lines.*

Chang has described smooth 3-folds of degree 12 that are birational to Calabi-Yau 3-folds (cf. [Ch], [DP]). Similar to the ones described in this paper they are bilinked to Fano 3-folds of degree 7, but they differ by their sectional genus. This difference is manifested in the appearance of non-normal singularities. The Calabi-Yau 3-folds of Theorem 8.3 do not deform to smooth ones: The double point formulas for 3-folds in \mathbf{P}^5 give the class of the non-normal singular locus in terms of the coefficients of the Hilbert polynomial, so any deformation also has non-normal singularities.

We find three open problems related to the topics of this paper particularly interesting:

Problem 1.1. Find H_6 -invariant non-normal but irreducible Calabi-Yau 3-folds, with a pencil of $(1, 6)$ -polarized abelian surfaces, degenerating to the union of two elliptic scrolls.

Problem 1.2. Consider the normalization of the Calabi-Yau 3-folds of degrees $10, \dots, 13$ of section 5. Find the invariants, the Betti- and Hodge numbers and describe the Kähler cone of these Calabi-Yau 3-folds.

Problem 1.3. Describe the elliptic scrolls and the reducible Calabi-Yau 3-folds in the cases $d = 14, 16, 18$.

We work over the complex numbers.

2. Numerical possibilities

Assume that an abelian surface $A \subset \mathbf{P}^n$ has two fibrations

$$p : A \rightarrow E \quad \text{and} \quad q : A \rightarrow F$$

in plane cubic curves. These fibrations define two \mathbf{P}^2 -scrolls. If the fibers of the maps p and q have intersection number ≥ 2 , then the planes in the two fibrations intersect in at least a line and the two \mathbf{P}^2 -scrolls coincide. So for our purposes we can assume that F and E are sections of p and q with $E \cdot F = 1$. In particular $A = E \times F$.

When there is no isogeny between E and F then A is the product of $E \times F$ in the Segre embedding of $\mathbf{P}^2 \times \mathbf{P}^2$ in \mathbf{P}^8 . We shall now assume that E is general in the sense that E has $\text{End}E \cong \mathbf{Z}$ and that $\gamma : E \rightarrow F$ is a primitive isogeny of degree l . Then $NS(A)$ is generated by E, F and Γ where Γ is the graph of γ . The numerical equivalence of a hyperplane divisor on A may therefore be expressed as

$$H \equiv aE + bF + c\Gamma, \quad a, b, c \in \mathbf{Z}.$$

Notice that these surfaces and these divisors really exist for any general elliptic curve E . We investigate for which numerical data they give us two plane cubic curve fibrations. The intersection numbers are given by the table:

	E	F	Γ
E	0	1	l
F	1	0	1
Γ	l	1	0.

With the requirements

$$H \cdot E = b + lc = 3, \quad H \cdot F = a + c = 3 \quad \text{and} \quad H \cdot \Gamma = al + b \geq 2,$$

we get

$$d/2 := H^2/2 = 9 - lc^2.$$

This means that $d/2 \leq 9$ and that there are the following possibilities for H :

Table 1.

d	l	c	a	b	$H \cdot \Gamma$
10	1	2	1	1	2
10	4	1	2	-1	7
12	3	1	2	0	6
14	2	1	2	1	5
16	1	1	2	2	4
18	*	0	3	3	*

The * in table 1 means that there is no isogeny between E and F involved.

Proposition 2.1. *There exist abelian surfaces with two plane curve fibrations of degree d in $\mathbf{P}^{\frac{d}{2}-1}$, when $d = 10, 12, 14, 16, 18$.*

Proof. To give examples it is now enough to check that H is very ample. For this we use Reider's criterion [Re], which in these cases reduces to check that there are no elliptic curves C on A with $H \cdot C \leq 2$. Any such curve C , not equivalent to E , F or Γ , must intersect each of these strictly positively, i.e.

$$C \cdot E > 0, \quad C \cdot F > 0, \quad \text{and} \quad C \cdot \Gamma > 0.$$

But in each case except the second $H = aE + bF + c\Gamma$ with $a + b + c \geq 3$ so $H \cdot C \geq 3$. In the second case $H = 2E - F + \Gamma$, so if $C = \alpha E + \beta F + \gamma \Gamma$, then

$H \cdot C < 3$ implies that

$$3\alpha + 3\beta + 7\gamma \leq 2.$$

The other inequalities above yield

$$\beta + 4\gamma \geq 1, \quad \alpha + \gamma \geq 1 \quad \text{and} \quad 4\alpha + \beta \geq 1,$$

while $C^2 = 0$, since C is elliptic, yields

$$\alpha\beta + 4\alpha\gamma + \beta\gamma = 0.$$

From the first four inequalities we get $\beta \leq -(\alpha + \gamma)$ which combines with the last equality to yield

$$0 = 4\alpha\gamma + \beta(\alpha + \gamma) \leq 4\alpha\gamma - (\alpha + \gamma)^2 = -(\alpha - \gamma)^2.$$

This only occurs when $\alpha = \gamma$, i.e. from the relation $C^2 = 0$, when $\alpha = 0$ or $2\alpha + \beta = 0$. The former is impossible since $\alpha + \gamma > 0$, while the latter is impossible since then $C \cdot H = 10\alpha - 6\alpha = 4\alpha \geq 4$.

In the first case $H \cdot \Gamma = 2$, so H is not very ample. In fact $|H|$ maps A two to one to a quintic elliptic scroll in \mathbf{P}^4 . In this case $E = F$ and the scroll is the symmetric product of E . Thus the two fibrations coincide in the image. In each of the other cases $|H|$ defines an embedding. \square

Remark 2.2. When $d = 18$ there is the simple example of

$$E \times F = E \times \mathbf{P}^2 \cap \mathbf{P}^2 \times F \subset \mathbf{P}^2 \times \mathbf{P}^2$$

in its Segre embedding in \mathbf{P}^8 . Clearly the union of the two scrolls deform in this case to Calabi-Yau 3-folds.

In this paper we shall concentrate on the case $d = 12$.

3. Elliptic scrolls and degenerate Calabi-Yau 3-folds in \mathbf{P}^5

From now on we consider abelian surfaces $A \subset \mathbf{P}^5$ of degree 12, i.e. with a $(1, 6)$ -polarization, and with two fibrations

$$p : A \rightarrow E \quad \text{and} \quad q : A \rightarrow F$$

in plane cubic curves. Furthermore we assume that E is general. We denote by X_E the scroll of planes of the fibration p , and by X_F the scroll of planes of the fibration q . The corresponding \mathbf{P}^2 -bundles are denoted V_E and V_F , respectively.

In the notation of the previous section and table 1 there is an isogeny

$$\gamma : E \rightarrow F$$

of degree 3. Furthermore, if Γ is the graph of the isogeny, then the hyperplane divisor is

$$H = 2E + \Gamma.$$

We fix an origin $o \in E$ and let $s_1 = \gamma(t_1)$, for some 3-torsion point t_1 on E , not in the kernel of γ . Let $h : F \rightarrow F$ be translation by s_1 , and let $\gamma^* : \text{Pic}^0 F \rightarrow \text{Pic}^0 E$ be the isogeny dual to γ . If $\mathcal{O}_F(o)$ is the line bundle of degree 1 whose unique section vanishes at o , then $h^*(\mathcal{O}_F(o)) \otimes \mathcal{O}_F(-o)$ generates the kernel of γ^* , i.e.

$$\ker \gamma^* = \{\mathcal{O}_F, h^*(\mathcal{O}_F(o)) \otimes \mathcal{O}_F(-o), (h^2)^*(\mathcal{O}_F(o)) \otimes \mathcal{O}_F(-o)\} \quad (h^3 = \text{id}).$$

Proposition 3.1. *The rank 3 vector bundle associated to the \mathbf{P}^2 -bundle V_F decomposes into $\mathcal{E} = \mathcal{L}_0 \oplus h^*\mathcal{L}_0 \oplus (h^2)^*\mathcal{L}_0$, where \mathcal{L}_0 is a line bundle of degree 2 on F and h is as above. Furthermore the scroll X_F is singular precisely along three lines, which span \mathbf{P}^5 .*

Proof. First, notice that Γ is contained in a pencil of hyperplanes defined by the pencil $|H - \Gamma| = |2E|$ on A . This pencil is the pullback by q of a divisor Δ_0 of degree 2 on F . Since any divisor in this pencil is a pair of plane cubic curves contained in a hyperplane, their planes intersect in a point. Thus the linear system $|\Delta_0|$ defines a morphism of F of degree 2 into the double locus of X_F .

Next we consider the translates Γ_t of Γ on A by a point t on E and find which Γ_t are contained in a pencil of hyperplanes. This happens precisely when $H - \Gamma_t$ is the pullback of a divisor of degree 2 from F , or equivalently when the restriction of $\Gamma - \Gamma_t$ to a fiber $E_f = q^{-1}(f)$ of q is trivial. But the translate Γ_t can be represented as

$$\{(x + t, \gamma(x)) | x, t \in E\}$$

so the intersection

$$(\Gamma - \Gamma_t) \cap E_f = \gamma^{-1}(f) - \gamma^{-1}(f) + 3t$$

which is trivial precisely when $3t = o$. Since γ is already an isogeny of degree 3, the 3-torsion points in the kernel of γ leave Γ invariant under translation, so in fact we may choose $t = t_1$ as above and we get 3 distinct translates of Γ which are contained in a pencil of hyperplanes in \mathbf{P}^5 . We denote them by $\Gamma_0 (= \Gamma), \Gamma_1, \Gamma_2$. They each span a \mathbf{P}^3 and determine a linear system of degree 2 on F , which we denote by $|\Delta_0|, |\Delta_1|, |\Delta_2|$. Each linear system $|\Delta_i|$ defines a morphism of degree 2 of F to the double locus of X_F . The images of these three maps are disjoint lines, and clearly the planes of X_F are spanned by the respective images by these three maps. Therefore V_F is defined by a decomposable rank 3 bundle \mathcal{E} of degree 6. If we denote the line bundle associated to the divisor Δ_i by \mathcal{L}_i , then

$$\mathcal{E} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2.$$

To check the differences between the line bundles \mathcal{L}_i , we consider the intersection

$$(\Gamma - \Gamma_t) \cap F_e = \gamma(e) - \gamma(e - t) = \gamma(t).$$

Since $3t_1 = o$, translation by $s_1 = \gamma(t_1)$ on F is a 3-torsion element that generates the kernel of the dual isogeny, $\gamma^* : \text{Pic}^0 F \rightarrow \text{Pic}^0 E$, i.e.

$$\mathcal{L}_i = (h^i)^* \mathcal{L}_0,$$

where h is the translation on F by s_1 .

It remains only to check the singularities of X_F . Any singular point of X_F is the intersection of two, possibly infinitely close, planes of X_F . But two planes intersect only if they span at most a hyperplane. The two planes are defined by a section of a line bundle $q^* \mathcal{L}$ for a line bundle \mathcal{L} of degree 2 on F . They intersect precisely when

$$\mathcal{E} \otimes \mathcal{L}^{-1}$$

has a section. But this is the case precisely when $\mathcal{L} = \mathcal{L}_i$ for $i \in \{0, 1, 2\}$. Furthermore, for each of these three cases $\mathcal{E} \otimes \mathcal{L}_i^{-1}$ has precisely one section, so the corresponding planes span a hyperplane and the intersection of the two planes is only a point. \square

The abelian surface A is an anticanonical divisor on V_F . We compute the sections of $-K_{V_F}$.

Lemma 3.2. $h^0(\mathcal{O}_{V_F}(-K_{V_F})) = 4$.

Proof. The natural isomorphism

$$H^0(\mathcal{O}_{V_F}(-K_{V_F})) \cong H^0(F, \text{Sym}^3 \mathcal{E} \otimes \mathcal{L}_0^{-1} \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1}),$$

reduces the computation to counting trivial summands of the rank 10 vector bundle

$$\text{Sym}^3 \mathcal{E} \otimes \mathcal{L}_0^{-1} \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1}.$$

Since $\mathcal{L}_0^{-1} \otimes \mathcal{L}_1^{-1} \otimes \mathcal{L}_2^{-1} = \mathcal{L}_0^{-3}$ this count is the number of summands in

$$\text{Sym}^3 \mathcal{E} = \text{Sym}^3(\mathcal{L}_0 \oplus h^* \mathcal{L}_0 \oplus (h^2)^* \mathcal{L}_0)$$

which equal \mathcal{L}_0^3 . As $h^3 = \text{id}$ this number is 4. \square

Lemma 3.3. *The \mathbf{P}^2 -bundle V_F is the quotient of a trivial bundle $\mathbf{P}^2 \times E$ by a cyclic group of order 3.*

Proof. Consider the isogeny $\gamma : E \rightarrow F$ again. Since $h^*(\mathcal{O}_F(o)) \otimes \mathcal{O}_F(-o)$ generates the kernel of γ^* the isogeny dual to γ ,

$$\gamma^* \mathcal{L}_0 \cong \gamma^* h^* \mathcal{L}_0 \cong \gamma^* (h^2)^* \mathcal{L}_0.$$

Therefore the pullback of $V_F = \mathbf{P}(\mathcal{E})$ over F via γ trivializes the bundle. The kernel of γ is a cyclic group of order three which acts on the pullback $\gamma^* \mathcal{E}$. \square

Thus we may construct V_F by starting with $\mathbf{P}^2 \times E$, and dividing by a suitable diagonal action of the cyclic group of order 3. For this we consider a vector space

$V = \langle x_0, x_1, x_2 \rangle$ with the action

$$\tau : x_i \mapsto \epsilon^i x_i, \quad i \in \mathbf{Z}_3.$$

As above, let $o \in E$ be the origin of E , and consider the linear system $|3o|$ on E . It embeds E as a plane cubic curve in \mathbf{P}^2 . We may choose $\langle e_0, e_1, e_2 \rangle$ as a basis for the underlying vector space of \mathbf{P}^2 such that

$$\tau : e_i \mapsto \epsilon^{-i} e_i$$

induces the action of translation by a 3-torsion point t_0 on E . The diagonal action defined by $\tau \in \mathbf{Z}_3$ on $V \times E$:

$$\tau : v \times e \mapsto \tau(v) \times e + t_0$$

acts without fixed points, so the quotient is a rank 3 vector bundle on $E/\langle t_0 \rangle = F$. The action of τ on V decomposes into the characters

$$V = \langle x_0 \rangle \oplus \langle x_1 \rangle \oplus \langle x_2 \rangle.$$

The anticanonical divisors on V_F pull back to anticanonical divisors on $\mathbf{P}^2 \times E$ which are invariant under the action of τ . But the anticanonical divisors on $\mathbf{P}^2 \times E$ are just the pullbacks of the cubic curves on the plane. The action of τ on the plane has the following basis of invariant cubics:

$$\langle x_0^3, x_1^3, x_2^3, x_0 x_1 x_2 \rangle.$$

Since these have no basepoints, there are no basepoints for the system of anticanonical divisors on V_F , and the general one is smooth. Notice furthermore that this linear system of invariant cubics contains the Hesse pencil

$$\langle x_0^3 + x_1^3 + x_2^3, x_0 x_1 x_2 \rangle,$$

and recall that the singular curves in this pencil are 4 triangles. In fact it is easy to check that these four triangles are the only triangles in the linear system of invariant cubics. The vertices of the triangle $x_0 x_1 x_2 = 0$ are mapped to the singular lines of X_F . The vertices of the three other triangles sweep out elliptic normal curves of degree 6 as we shall see next.

Proposition 3.4. *The scroll X_F is the 3-torsion translation scroll of an elliptic normal curve in \mathbf{P}^5 .*

Proof. Consider an elliptic normal curve C of degree 6 in \mathbf{P}^5 , embedded by the linear system $|6o|$. For any $P \in C$ consider the translation scroll

$$V_P = \cup_{y \in C} \langle y, y + P, y + 2P \rangle.$$

This is, for general P , a \mathbf{P}^2 -scroll of degree 18. When $3P = o$ the points y , $y + P$ and $y + 2P$ generate the same plane, so then the translation scroll has degree 6. In this case

$$\langle y, y + P, y + 2P \rangle \cap \langle z, z + P, z + 2P \rangle \neq \emptyset$$

precisely when $3y + 3z = o$. But then the pencil $|y + z|$ defines a map from C to the double locus of V_P . Now $3(y + z) = 3(y + z + P) = 3(y + z + 2P)$, so this map factors through the isogeny $C \rightarrow C/\langle P \rangle$. Thus the 9 linear systems $|y + z|$ with $3(y + z) = 0$ define three pencils of pairs of planes each defining a double line for the translation scroll. The translation scroll is clearly a scroll over $C/\langle P \rangle$ and its associated vector bundle decomposes into the sum of three line bundles of degree 2 which define the three double lines. The differences between any two of these line bundles are the powers of some 3-torsion line bundle. If $C \cong E$ and $C/\langle P \rangle = F$, we have $V_P \cong X_F$. \square

Finally, we consider the union of the two elliptic scrolls $X_1 = X_E$ and $X_2 = X_F$ in \mathbf{P}^5 which both contain the $(1, 6)$ -polarized abelian surface A . Let $V_1 = V_E$ and $V_2 = V_F$ be the corresponding \mathbf{P}^2 -bundles. Then V_1 and V_2 are the normalizations of X_1 and X_2 and $A \in |-K_{V_1}|$, resp. $A \in |-K_{V_2}|$. Let $Y = X_1 \cup X_2$.

Proposition 3.5. *Y has a partial desingularization $Y_0 = V_1 \cup V_2$, which is a Calabi-Yau 3-fold, i.e. $K_{Y_0} = \mathcal{O}_{Y_0}$ and $q = h^1(\mathcal{O}_{Y_0}) = 0$. In particular $V_1 \cap V_2 = A$.*

Proof. First, we may use the previous notation and let $A = E \times F$ with hyperplane divisor $H = 2E + \Gamma$.

Notice that the abelian surface A does not intersect the singular lines of X_F . In fact $|H - E_f|$, for a plane cubic curve E_f passing through a singular point, would then have a basepoint. This is impossible by Reider's criterion [Re].

We show next that X_1 and X_2 intersect transversally along A : Near A the 3-fold Y is the intersection of two smooth 3-folds. If the intersection is not transversal the tangent spaces of the two scrolls at some point coincide. This is possible only if the planes of the two scrolls at the given point intersect along a line. Those two planes intersect A in elliptic cubic curves E_f and F_e which meet in a point. Any curve in $|H - E_f - F_e|$ has degree 6 and arithmetic genus 2. Since there is a pencil of hyperplanes through the two planes, this curve moves in a pencil on A , this is impossible by Riemann Roch, so transversality follows. If $X_1 \cap X_2$ contains a point disjoint from A , then a plane in X_1 meets a plane in X_2 along a line and the argument above applies again. Thus X_1 and X_2 meet transversally along A .

Now, look at the partial desingularization Y_0 of Y obtained by normalizing X_1 and X_2 . Thus we may write $Y_0 = V_1 \cup V_2$ and $V_1 \cap V_2 = A$. To show that $q(Y_0) = 0$ we consider $\text{Pic}^0 Y_0$. An element in $\text{Pic}^0 Y_0$ is a pair $(\mathcal{M}_1, \mathcal{M}_2)$ with $\mathcal{M}_i \in \text{Pic}^0 V_i$ which glue along A . Consider the fibrations

$$p_1 : V_1 \rightarrow F \quad \text{and} \quad p_2 : V_2 \rightarrow E.$$

Now, $\mathcal{M}_i = p_i^*(\mathcal{N}_i)$, where $\mathcal{N}_1 \in \text{Pic}^0(F)$ resp. $\mathcal{N}_2 \in \text{Pic}^0(E)$. On A we have sections of each p_i which are fibres of the opposite map. We identify these sections with F and E respectively. Then $\mathcal{M}_1|_E = \mathcal{O}_E$ and $\mathcal{M}_2|_F = \mathcal{O}_F$. So if we want to glue \mathcal{M}_1 and \mathcal{M}_2 we must have that

$$\mathcal{O}_E = \mathcal{M}_1|_E = \mathcal{N}_2$$

and

$$\mathcal{O}_F = \mathcal{M}_2|_F = \mathcal{N}_1,$$

and hence $(\mathcal{M}_1, \mathcal{M}_2) = (\mathcal{O}_{X_1}, \mathcal{O}_{X_2})$. This shows that $\text{Pic}^0 Y_0 = \{\mathcal{O}_{Y_0}\}$. Since $\text{Pic}^0 Y_0$ is a reduced group scheme in characteristic 0 and since $H^1(X, \mathcal{O}_{Y_0})$ is the tangent space at the origin, it follows that $q(Y_0) = h^1(Y_0, \mathcal{O}_{Y_0}) = 0$.

Since $A \in |-K_{V_1}|$, resp. $A \in |-K_{V_2}|$ it is clear that the restriction of K_{Y_0} to V_1 and to V_2 is trivial. But $\text{Pic}^0 Y_0 = \{\mathcal{O}_{Y_0}\}$, so $K_{Y_0} = \mathcal{O}_{Y_0}$. \square

The 3-fold $Y = X_E \cup X_F$ is non-normal, singular along six lines three on each scroll in addition to the surface $A = X_E \cap X_F$. In the next section we describe a series of non-normal Del Pezzo 3-folds. In section 5 we show that these are bilinked to non-normal Calabi-Yau 3-folds. After some further analysis of the equations of Y in sections 6 and 7 we show in section 8 that Y is a degeneration of these non-normal Calabi-Yau 3-folds.

4. Non-normal Del Pezzo 3-folds in \mathbf{P}^5

Del Pezzo 3-folds are 3-folds W for which $K_W \cong -2H$, where H is ample. Accordingly any smooth surface on W in $|H|$ is a Del Pezzo surface.

Let $V_t \subset \mathbf{P}^{t+1}$ with $t = 3, \dots, 8$ be the image of \mathbf{P}^3 by the map defined by all quadrics through $8 - t$ general points in \mathbf{P}^3 . Then V_t is a Del-Pezzo 3-fold. We describe the image $W_t \subset \mathbf{P}^5$ of the general projection of V_t .

In particular we want to describe the singular non-normal locus. Thus we are interested in the cases $t = 5, 6, 7, 8$ and will prove

Theorem 4.1. *W_t is non-normal along $\binom{t-3}{2}$ skew lines and has $\binom{8-t}{2}$ additional ordinary double points when $t = 5, 6, 7, 8$.*

To prove this we will use a result of Reye on linear systems of quadrics. To explain Reye's result we need the notion of apolarity applied to quadrics in \mathbf{P}^3 . Thus let $S = k[x_0, \dots, x_3]$ and $T = k[y_0, \dots, y_3]$, and define a pairing $S_2 \times T_2 \rightarrow k$ by letting S operate as differential operators on T and vice versa.

We say that quadrics in S and T are apolar if they are orthogonal with respect to this pairing. In our situation we think of S as the coordinate ring of \mathbf{P}^3 and T as the coordinate ring of \mathbf{P}^3 . Starting with a 6-dimensional subspace of quadrics $V \subset S_2$, there is a 4-dimensional subspace i.e. a web of quadrics $V^\perp \subset T_2$.

Since any quadric in V is apolar to any member of V^\perp , we say that V and V^\perp are apolar sets of quadrics. Consider the discriminant D of the space V^\perp of quadrics. This is a quartic surface defined by the determinant of a symmetric 4×4 matrix with linear entries. The quadrics in V^\perp of rank 1 and 2 are respectively triple and double points on D . The possible numbers of rank 1 and rank 2 quadrics are given in the following table:

Table 2.

rank 1 quadrics	rank 2 quadrics
4	$0/\infty$
3	1
2	3
1	6
0	10

This follows from a few lemmas which have independent interest for us.

Lemma 4.2. *Each rank 1 quadric in V^\perp determines a basepoint for the quadrics in V and vice versa. Each rank 2 quadric in V^\perp determines a line contained in V and vice versa. Alternatively, if ρ is the map defined by the linear system V of quadrics, then each rank 2 quadric in V^\perp determines a line in the source double point locus of ρ and vice versa.*

Proof. Note that if $a = (a_0, a_1, a_2, a_3)$, $L = \sum_{i=0}^3 a_i y_i$ and $q \in S_2$, then

$$L^2(q) = 2q(a).$$

Now each rank 1 quadric in V^\perp has the form L^2 for some point a , and apolarity says that $L^2(q) = 2q(a) = 0$ for every quadric q in V so a is a basepoint for V . Conversely, if $a = (a_0, a_1, a_2, a_3)$ is a basepoint for V and $L = \sum_{i=0}^3 a_i y_i$, then $L^2(q) = 2q(a) = 0$ for every $q \in V$ so $L^2 \in V^\perp$.

Each rank 2 quadric in V^\perp has the form $L_1^2 + L_2^2$ for some linear forms $L_1 = \sum_{i=0}^3 a_i y_i$ and $L_2 = \sum_{i=0}^3 b_i y_i$. Let $l \subset \mathbf{P}^3$ be the line spanned by the points $a = (a_0, a_1, a_2, a_3)$ and $b = (b_0, b_1, b_2, b_3)$. Let $V_l \subset V$ be the subspace of quadrics which vanish on l . Then

$$V_l = \{q \in V \mid L_1^2(q) = 2q(a) = L_2^2(q) = 2q(b) = L_1 L_2(q) = 0\}.$$

But $(L_1^2 + L_2^2)(q) = 0$ for every $q \in V$, so V_l has codimension 2 in V . Conversely if V_l has codimension 2 in V , then some linear combination of L_1^2, L_2^2 and $L_1 L_2$ is contained in V^\perp . But any such linear combination is a rank 2 (or rank 1) quadric, and the lemma follows. \square

Porteous' formula [cf. Fu 14.4.11] computes the number of rank 2 quadrics in a general web of rank 4 quadrics. This number is 10. Reye found a geometric interpretation of these 10 rank 2 quadrics considering apolar twisted cubic curves to the web of quadrics, i.e. twisted cubic curves whose defining net of quadrics is apolar to V^\perp .

By a determinantal net of quadrics we mean a net (i.e. a 3-dimensional space) of quadrics which is generated by the 2×2 minors of a 2×3 matrix of linear forms. The general determinantal net of quadrics generates the ideal of a twisted cubic curve.

Lemma 4.3. (Reye). *The general 6-dimensional subspace $V \subset S_2$ contains precisely two determinantal nets of quadrics, which together span V . If C_1 and C_2 are the twisted cubic curves defined by these two nets, then every rank 1 quadric in V^\perp corresponds to a point of intersection between C_1 and C_2 and vice versa. Furthermore every rank 2 quadric in V^\perp corresponds to a common secant line for C_1 and C_2 and vice versa.*

Proof. The number of determinantal nets of quadrics in a general 6-dimensional space V of quadrics is nowadays computable by quantum cohomology [Kre] (compute the number of twisted cubic surface scrolls through nine points in \mathbf{P}^4 in the quantum cohomology of the Grassmannian of lines and intersect with a general \mathbf{P}^3), a few years ago by modern intersection theory [ES] and in the ancient times by direct geometric arguments [Rey]. We leave the choice of reference to the reader.

Given two determinantal nets which span V , the correspondence between points of intersection and the rank 1 quadrics follows from Lemma 4.2.

For the second correspondence consider first a common secant line to C_1 and C_2 . This line and any one of the two curves form a complete intersection $(2, 2)$. Therefore the line lies in a pencil of quadrics from each of the two determinantal nets. Together the two pencils form a web of quadrics in V which by Lemma 4.2 corresponds to a rank 2 quadric in V^\perp . On the other hand the secant lines to a twisted cubic curve form a congruence of bidegree $(1, 3)$ in the Grassmannian of lines in \mathbf{P}^3 . Thus two general twisted cubic curves have

$$(1, 3) \cdot (1, 3) = 1 + 9 = 10$$

common secant lines. This is exactly the number of rank 2 quadrics in the web V^\perp , so the second correspondence follows. \square

To fill in the remainder of table 2 we want to compute how much the number of rank 2 quadrics decreases when the web acquires a rank 1 quadric. We give an argument using Reye's geometric interpretation. Our arguments will depend on a genericity assumption, i.e. the space V is general with a given number of basepoints. The argument would go through without this assumption also, but then the numbers in table 2 would have different interpretations. Since we will only use general systems V we do not consider the degenerate cases.

When the space V of quadrics have basepoints, then there will be infinitely many apolar twisted cubic curves to V , but taking two of them will always suffice for our argument. First note that as long as the two twisted cubic curves have less than 4 common points, the corresponding nets of quadrics do not intersect.

When the web contains one rank 1 quadric, then the two twisted cubic curves have one common point. The number of common secant lines passing through this point is, with our genericity assumption, easily computed by projection from the point, it is 4, the number of intersection points between two conics in the plane. So the web has 6 rank 2 quadrics in addition to the rank 1 quadric. This is the second row in the table.

When the web contains 2 rank 1 quadrics, then the two twisted cubic curves have two common points. In this case there are 4 common secant lines through each of the two intersection points, and one of these is the line passing through the two points, so there are exactly 3 common secant lines which do not pass through any of the two common points. Thus the web has 3 rank 2 quadrics in addition to the two rank 1 quadrics. This is the third row in the table.

When the web contains 3 rank 1 quadrics, the two twisted cubic curves have 3 points in common. There are 4 common secant lines through each intersection point, and adding up three are counted twice, so we get only one common secant line which does not pass through any of the intersection points. Thus the web has one rank 2 quadric in addition to the three rank 1 quadrics. This is the fourth row in the table.

When the web has more than 3 rank 1 quadrics, the twisted cubics have at least 4 points of intersection. In this case the number of common secant lines that does not pass through the intersection points is infinite or zero depending on whether the two determinantal nets intersect or not. This covers the remaining row in the table.

Lemma 4.4. *Consider the map ρ defined by the linear system V of quadrics, and consider a subscheme Z of length 2 which does not intersect the baselocus of V . Then Z is mapped to a point by ρ if and only if either the restriction of ρ to the unique line passing through Z is $2 : 1$ onto a line, or this line contains two base points for V .*

Proof. The linear system V restricted to the line through Z has degree 2. If this linear system has no basepoints, then the image of the line by ρ is a line or a conic section. It is a line if and only if some subscheme of length 2 is mapped to a point. If the line through Z intersects the baselocus, the intersection must contain two points, such that the line is contracted by V . \square

Proof of Theorem 4.1. Combining Lemmas 4.2, 4.3 and 4.4 we find that a general 6-dimensional linear system of quadrics with $8 - t$ basepoints and $t = 5, 6, 7, 8$ defines a rational map from \mathbf{P}^3 to \mathbf{P}^5 , whose image has degree t and whose non-normal double locus consists of $\binom{t-3}{2}$ disjoint lines. Any line between basepoints is contracted to an isolated singularity. The number of isolated singular points is therefore $\binom{8-t}{2}$. It is easily checked by restriction to general hyperplane sections that these singular points are ordinary double points. \square

By abuse of notation we call the varieties W_t Del Pezzo 3-folds. Their normalization have only isolated double points from the contracted lines between basepoints, these occur when $t \leq 6$. For each t we want to describe the ideal of W_t and understand their linkage class.

Proposition 4.5. *A 3-fold W_5 is bilinked to the union of two \mathbf{P}^3 's which span \mathbf{P}^5 . A 3-fold W_6 is bilinked to the union of three \mathbf{P}^3 's. The ideals of general Del*

Pezzo 3-folds W_7 and W_8 are generated by quartics and quintics, and quintics and sextics respectively.

Proof. To understand the ideal of W_t we first investigate the ideal of the singular lines. For $t = 5$ there is one line so this case is trivial. The case $t = 6$ is also easy since there are three lines; they span \mathbf{P}^5 as soon as the projection is general, as is easily verified. For $t = 7$ and $t = 8$ the situation is a bit more involved.

Lemma 4.6. *The 6 singular lines of a general Del Pezzo 3-fold W_7 lie in a determinantal net of quadrics in \mathbf{P}^5 , in fact in the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^2$.*

Proof. Let V be the linear system of quadrics which define the rational map of \mathbf{P}^3 onto W_7 . By Lemma 4.3 the linear system V contains and is spanned by at least two determinantal nets. Of course, the corresponding twisted cubic curves pass through the basepoint, so the union of any two is rational and is therefore contained in a cubic surface. By genericity we may assume that this surface is smooth. On this cubic surface there is a pencil of twisted cubic curves through the basepoint with six common secant lines. This pencil of curves corresponds to a linear pencil of nets of quadrics, so since two nets are contained in V they all are. The images of these twisted cubic curves on W_7 are plane curves. Since the twisted cubic curves pass through the basepoint of V these plane curves have degree 5. The 6 common secant lines are mapped to the 6 singular lines on W_7 and they account for the 6 singular points on each of these plane rational quintic curves.

Consider the space of quadrics passing through the 6 singular lines. If any of these quadrics intersects a plane of the plane quintic curves properly, its intersection would be a conic section passing through the 6 singular points of the quintic curve. But this is impossible by Bezout. Therefore any quadric which passes through the 6 lines must contain the planes of these curves. The intersection of these quadrics is therefore at least a threefold. Now, 6 lines impose at most 18 conditions on quadrics, so there are at least 3 such quadrics and the intersection is a threefold. A codimension 2 variety contained in 3 quadrics is a rational cubic scroll. If this scroll is singular, it is a cone, and any two planes meet. In this case two planes span a hyperplane. Pulled back to the cubic surface of the twisted cubic curves, this hyperplane corresponds to a quadric which contains the two curves. But the union of the two curves has arithmetic genus 0, so it is not contained in a quadric and a cubic surface. Therefore the scroll is smooth, isomorphic to the Segre 3-fold scroll. \square

Proposition 4.7. *W_7 is contained in five quartic hypersurfaces, they define an arithmetically Cohen Macaulay 3-fold, the union of W_7 and the rational cubic scroll R which contains the singular lines of W_7 . The quartic hypersurfaces are defined by the maximal minors of a 4×5 matrix with linear entries.*

Proof. Again let V be the linear system of quadrics which define the rational map of \mathbf{P}^3 onto W_7 . In the proof of Lemma 4.6 we saw that there is a pencil of twisted

cubic curves through the basepoint p of V whose defining nets of quadrics are contained in V . These twisted cubic curves sweep out a cubic surface S_3 whose image contains the singular lines on W_7 and is contained in the rational cubic scroll R . The linear system of quadrics V restricts to S_3 with one base point, so the image has degree 11. Consider the quartic surfaces through S_3 and singular in p ; they consist of the union of S_3 and planes through p , so these quartics form a net. On the other hand, the image of S_3 in \mathbf{P}^5 is contained in the net of quadrics through R , and these quadrics pulled back to \mathbf{P}^3 , correspond to quartic surfaces through S_3 singular at p , i.e. to the above net of quartics. Since the net of quartics has no unassigned basepoints, the quadrics through R define precisely the image of S_3 on W_7 , i.e. the intersection $W_7 \cap R$ is precisely the image of the surface S_3 . The union $W_7 \cup R$ has degree 10 and genus 11. To conclude that the union is arithmetically Cohen Macaulay we give an example. Let V be the space of quadrics

$$\langle x_0^2 + x_1x_2, x_1^2 + x_2x_3, x_2^2 + x_3x_0, x_0x_1, x_0x_1 + x_2x_3, x_0x_2 + x_1x_3 \rangle.$$

A straightforward computation in [MAC] shows that the Del Pezzo 3-fold W_7 defined by V lies in precisely 5 quartics, the 4×4 minors of a 4×5 matrix with linear entries, i.e. these quartics define an arithmetically Cohen Macaulay scheme of degree 10 and genus 11. Since this is an open condition in the Hilbert scheme [Ell], the same is true for the general projection W_7 . \square

For W_8 we get somewhat less.

Lemma 4.8. *W_8 contains two plane rational sextic curves, and each of the singular lines is spanned by a pair of nodes of these two sextic curves. In particular there are sextic generators in the ideal of W_8 .*

Proof. In this case there are two apolar twisted cubic curves, these are mapped to plane sextic curves with 10 double points at the intersection of these planes with the 10 singular lines of W_8 . So we need sextic generators in the ideal of W_8 . \square

We are now ready to give some numerical results for the ideals of general Del Pezzo 3-folds W_t of degree $5 \leq t \leq 8$.

Lemma 4.9. *Table 3 gives the degrees d and the number of generators in the ideal of W_t , for $t = 5, 6, 7, 8$.*

Table 3.

	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$
$t = 5$	0	5	0	0	0
$t = 6$	0	1	7	0	0
$t = 7$	0	0	5	5	0
$t = 8$	0	0	1	10	≥ 1

Proof. The following spaces V of quadrics

$$\begin{aligned} &\langle x_1x_2 + x_0x_3, x_3^2 + x_2x_0, x_3^2 + x_1x_0, x_3^2 + x_2x_1, x_0x_1 + x_2x_3, x_0x_2 + x_1x_3 \rangle, \\ &\langle x_0^2 + x_1x_2, x_1^2 + x_2x_3, x_3x_0, x_0x_1, x_0x_1 + x_2x_3, x_0x_2 + x_1x_3 \rangle, \\ &\langle x_0^2 + x_1x_2, x_1^2 + x_2x_3, x_2^2 + x_3x_0, x_0x_1, x_0x_1 + x_2x_3, x_0x_2 + x_1x_3 \rangle, \end{aligned}$$

and

$$\langle x_0^2 + x_1x_2, x_1^2 + x_2x_3, x_2^2 + x_3x_0, x_3^2 + x_0x_1, x_0x_1 + x_2x_3, x_0x_2 + x_1x_3 \rangle,$$

have respectively 3,2,1 and no basepoints. The ideals of the corresponding Del Pezzo 3-folds in \mathbf{P}^5 are easily computed in [MAC] and have Betti numbers as in the table.

For the proof of the lemma we show that the table represents a lower bound on the number of generators in each of the given degrees. The lemma then follows by semicontinuity.

In each case we define Σ to be the union of the singular lines L_i on W_t , i.e. precisely the non-normal double point locus of W_t . Let V_t be the normalization of W_t , i.e. V_t is isomorphic to \mathbf{P}^3 blown up in $8 - t$ points, and with $\binom{8-t}{2}$ lines contracted to ordinary double points. The map $\varphi : V_t \rightarrow W_t$ is double precisely along Σ , thus we get an exact sequence of sheaves

$$0 \rightarrow \mathcal{O}_{W_t} \rightarrow \varphi_* \mathcal{O}_{V_t} \rightarrow \mathcal{O}_{\Sigma}(-1) \rightarrow 0,$$

where $\mathcal{O}_{\Sigma} = \oplus_{i=1}^l \mathcal{O}_{L_i}$ and $l = \binom{t-3}{2}$. Since V_t is a Del Pezzo 3-fold, the cohomology of this sequence gives $h^3(\mathcal{O}_{W_t}(k)) = h^3(\varphi_* \mathcal{O}_{V_t}(k)) = 0$, when $k \geq 3$. Furthermore the Euler characteristic of the relevant twists of the ideal sheaf of W_t is easily computed from this exact sequence together with the exact sequence

$$0 \rightarrow \mathcal{I}_{W_t}(k) \rightarrow \mathcal{O}_{\mathbf{P}^5}(k) \rightarrow \mathcal{O}_{W_t}(k) \rightarrow 0.$$

We collect the results in the following table.

Table 4.

	k	$\chi(\mathcal{O}_{W_t}(k))$	$h^0(\varphi_* \mathcal{O}_{V_t}(k))$	$h^0(\mathcal{O}_{\Sigma}(k-1))$	$h^0(\mathcal{O}_{\mathbf{P}^5}(k))$	$\chi(\mathcal{I}_{W_t}(k))$
$t = 5$						
	3	51	54	3	56	5
$t = 6$						
	3	55	64	9	56	1
	4	113	125	12	126	13
$t = 7$						
	4	121	145	24	126	5
	5	221	251	30	252	31
$t = 8$						
	4	125	165	40	126	1
	5	236	286	50	252	16

By restriction to general \mathbf{P}^3 sections of W_t it is easy to check that $h^1(\mathcal{O}_{W_t}(k)) = 0$ for the values of k in the table. Since additionally $h^3(\mathcal{O}_{W_t}(k)) = 0$ when $k \geq 3$, the Euler characteristic of the twisted ideal is a lower bound for $h^0(\mathcal{I}_{W_t}(k))$. Thus the Betti numbers of the ideal follow except in the case of quintics and W_7 and sextics and W_8 . These cases are accounted for in Proposition 4.7 and Lemma 4.8. In fact, from Lemma 4.7 we get 4 linear syzygies among the 5 quartics in the ideal of W_7 . Therefore there are also 4 extra quintic generators in the ideal, 5 altogether. Lemma 4.8 says that there are sextic generators in the ideal of W_8 . \square

To finish the proof of Proposition 4.5 it remains to consider the linkage classes of W_5 and W_6 . The 3-fold W_5 is linked $(3, 3)$ to a rational 3-fold scroll of degree 4. This lies in a quadric and is linked $(2, 3)$ to two \mathbf{P}^3 's, which clearly is minimal in its even biliaison class.

W_6 is linked $(3, 4)$ to a 3-fold U with sectional genus 1 which lies on two cubics. Consider a \mathbf{P}^3 spanned by two singular lines in W_6 . It intersects W_6 in the two lines and in two additional skew lines each intersecting both the singular lines. Clearly every cubic through W_6 contain this \mathbf{P}^3 , so U must intersect it in a quartic surface singular along the two singular lines. U is therefore an elliptic scroll linked $(3, 3)$ to the union of three \mathbf{P}^3 's (cf. also section 6). This concludes the proof of Proposition 4.5. \square

In the last section we need a converse to Proposition 4.7.

Proposition 4.10. *Let W be a 3-fold with sectional genus 1, with non-normal double points along 6 skew lines, no 3 in a \mathbf{P}^3 and not all 6 on a rational normal quartic scroll. Assume that there is a Segre cubic scroll R in \mathbf{P}^5 containing the 6 skew lines transverse to its planes and intersecting W in a surface of degree 11, linked to 4 planes in the intersection of R with a quintic hypersurface. Assume furthermore that $W \cup R$ is scheme-theoretically defined by the 4×4 minors of a 4×5 matrix with linear entries, and that the only common singularities of these quartics are the 6 lines. Then W is a Del Pezzo 3-fold W_7 .*

Proof. Consider the 4×5 matrix M whose maximal minors define $R \cup W$. Let $\langle z_0, \dots, z_3 \rangle$ be the coordinates of \mathbf{P}^3 , then the 5 bilinear equations

$$(z_0, z_1, z_2, z_3) \cdot M = (0, 0, 0, 0, 0)$$

define a 3-fold T in $\mathbf{P}^3 \times \mathbf{P}^5$. The projection to \mathbf{P}^5 is onto $W \cup R$, while the other projection is onto \mathbf{P}^3 . The fiber in T over any point in \mathbf{P}^3 is linear, defined by the 5 linear equations in the 6 coordinate functions of \mathbf{P}^5 , so for the general point in \mathbf{P}^3 the fiber is a single point. The fiber over any point in $W \cup R$ is also linear, with dimension equal to 3 minus the rank of M at the target point. The points where M has rank at most 2, are singular on $W \cup R$, in fact they are singular on any quartic minor of M , so by our assumption, these points all lie on the 6 singular lines. Outside the 6 singular lines M has rank 3 and therefore there is a rational

map

$$\varphi : W \cup R \dashrightarrow \mathbf{P}^3$$

which is a morphism outside these 6 lines. This map is birational on one component of $W \cup R$ and contracts every other component to a surface, a curve or a point. The inverse map

$$\psi : \mathbf{P}^3 \dashrightarrow W \cup R$$

is defined by the maximal minors of a 5×6 -matrix M' with linear entries. M' is obtained from M by interchanging rows and linear forms. Surfaces that are images of components of $W \cup R$ are fixed components for the linear system of quintic minors of M' .

Our first aim is to show that W has to be irreducible. We analyse carefully possible reducible components of the surface $S = W \cap R$. For this we start with the intersection of S with a general plane in R .

On R the surface $S \equiv 5h - 4f$, where h is the class of a hyperplane section, while f is the class of a plane. Since W is non-normal along 6 lines transverse to the planes of R , the intersection of S with a general plane is a curve C of degree 5 singular in 6 points, the points of intersection between the plane and the singular lines. Furthermore, no three singular lines span a \mathbf{P}^3 , so no three of the singular points of C are collinear. More generally, any effective divisor of type $ah - af$ is the product of a plane curve of degree a and \mathbf{P}^1 . Geometrically each \mathbf{P}^1 is a line transverse to all planes in R . Conversely, any irreducible divisor of type $h + bf$ that contains more than one line transverse to all planes has $b = -1$ and is a quadric surface, while any irreducible divisor of type $2h + bf$ that contains more than 4 lines transverse to all planes has $b = -2$ and is a rational normal quartic scroll. In our situation, this means that three singular lines intersect no plane in collinear points. Also, since not all 6 singular lines are on a rational normal quartic scroll, the 6 singular points cannot lie on a conic section. Therefore, if C is reducible it is either the union of 2 conics and a line, or the union of a conic and a cubic. In the former case the singular points are the points of intersection between the two conics and one point of intersection between the line and each conic, while in the latter case 5 singular points are points of intersection between the conic and the cubic, while the last point is singular on the cubic.

Next, the surface S intersects each general line in R transverse to the planes in one point. Since the intersection of S with a general plane has at most 3 components, S itself has at most 4 components: Only the planes in R do not intersect the general plane in a curve, so since exactly one component intersects every line transverse to the planes, there are at most 4 components of S .

To describe the possible components of S we first note that on R any effective divisor is of type $ah - bf$, with $a \geq 0$ and $b \leq a$. In our two cases of possible reducible curves in general planes we have the following possible decompositions of S into irreducible components:

$$S \equiv 5h - 4f = b_0f + (h - b_1f) + (2h - b_2f) + (2h - b'_2f)$$

where $b_1 + b_2 + b'_2 - b_0 = 4$ and

$$S \equiv 5h - 4f = b_0f + (2h - b_2f) + (3h - b_3f)$$

where $b_2 + b_3 - b_0 = 4$. With the above restriction on the b_i , the former case occurs only when $(b_0, b_1, b_2, b'_2) = (1, 1, 2, 2)$, $(b_0, b_1, b_2, b'_2) = (0, 0, 2, 2)$ or $(b_0, b_1, b_2, b'_2) = (0, 1, 2, 1)$. The latter case occurs only when $(b_0, b_2, b_3) = (1, 2, 3)$, $(b_0, b_2, b_3) = (0, 2, 2)$ or $(b_0, b_2, b_3) = (0, 1, 3)$.

We are now ready to analyse possible components of W . Clearly any two components of W intersect each other along a surface or a curve (or both). If two components of $W \cup R$ intersect each other along a curve, then every hypersurface through $W \cup R$ has to be singular along this curve, so by our assumption this happens only along the 6 singular lines. The sectional genus of W is 1, thus for the general \mathbf{P}^3 section of W the arithmetic genus is 1 plus the number of intersection points with the singular lines. This means that for a general \mathbf{P}^3 passing through a plane of R , the contribution to the arithmetic genus of the plane curve C is 0. The degree of W is 7, so residual to C in this intersection is a curve of degree 2. It must intersect C in 2 points, and it must itself have arithmetic genus 0, so it is a conic section, two connected lines or a double line. This very much restricts the possibilities for the irreducible components of W .

In our analysis of the components W_d of W , we shall index them by their degree, unless the degree is not specified, in which case we use index 0. This notation applies only in this proof and should not be confused with Del-Pezzo 3-folds W_t . A component W_0 of W must intersect R in a curve or a surface. If it intersects R only in a curve, then this curve meets the general plane in R only in points, thus this component intersects a general \mathbf{P}^3 through a general plane in R in a curve not contained in the plane. Therefore this component has degree 1 or 2. Similarly if W_0 intersects the general plane in R in a curve of degree a , then the degree of the component is a , $a + 1$ or $a + 2$. In fact, if W_0 spans \mathbf{P}^5 , i.e. when $a \geq 2$, then the degree is strictly greater than a . Of course, while W_0 is irreducible, $W_0 \cap R$ may still be reducible. Anyway, we may enumerate the different cases in a table, where the columns are ordered by the degree of the curve of intersection between a component and a general plane, while the entries are the degrees of the components themselves:

Table 5.

0	1	2	3	4	5
1	0	0	0	0	6
1	1	0	0	5	0
0	1	0	0	0	6
0	1	3, 3	0	0	0
0	0	3	4	0	0
0	0	0	0	0	7

In the first two cases of table 5 the first linear component W_1 may intersect R in a plane and a line, or in a connected curve of degree 3. The latter may be

excluded since no connected cubic curve is supported on the 6 singular lines. In the former case the line must be one of the 6 singular lines. The remaining part of W is a component of degree 6 or two components of degree 1 and 5, respectively. These remaining components must intersect R in a surface of type $5h - 5f$, i.e. they do not intersect the general line in R transverse to the planes. In the first case the linear component intersects the component W_6 of degree 6 along a quadric surface, since the sectional genus is 1. The linear component W_1 intersects R in a plane, but W_6 intersects the general plane in R in a curve of degree 5, so this plane must also be contained in W_6 . This contradicts the fact that W_6 does not intersect the general line in R transverse to the planes. The second case is impossible for the same reason.

In the third case of the table there is a linear component W_1 that intersects R along a quadric surface and a component W_6 that intersects R in a surface of type $4h - 3f$. Again the two components have also to intersect along a quadric surface. The intersection of the two components on R is a curve of type $(h - f)(4h - 3f) = 4h^2 - 7hf$, i.e. a curve of degree 5. On the other hand on W_1 each of the two other components intersect along a quadric surface and two singular lines. This adds up to a curve of degree 6 in the intersection of all three components. This is a contradiction. The fourth case is entirely similar.

In the fifth case there are two components W_3 and W_4 of degree 3 and 4 respectively. They intersect each other in a surface of degree 2. Furthermore they intersect on R in a curve of type $(2h - f)(3h - 2f) = 6h^2 - 7hf$, i.e. a curve of degree 11. The union of the 5 singular lines in this intersection is a curve of type $5h^2 - 10hf$, so the remaining part is of type $h^2 + 3hf$. This is a curve of degree 6. Again the surface $W_3 \cap W_4$ has degree 2. If it is irreducible, then the lines of one of the rulings are at least 3-secants to R , impossible, so the quadric surface is contained in R , again contradicting our assumption. If the surface $W_3 \cap W_4$ is the union of two planes, then at least one of the planes intersects R in a curve of degree at least 3, again contradicting our assumption, so this case is also impossible. This concludes the proof that W is irreducible.

It remains to show that R is contracted by φ . If not, W must be contracted. First of all this is possible only if W is swept out by lines, i.e. has a line through every point. For this consider a general plane in R . It intersects W in a plane quintic curve with 6 double points. W has degree 7, so in a general \mathbf{P}^3 through the plane W has a curve of degree 2 residual to the plane curve. Since the arithmetic genus of W is 1 and the plane curve is rational, this residual curve is a conic or two connected lines. Now, W can only have a line through each point if this residual curve is always two lines. On the other hand if these two lines are contracted by φ , they are contracted to the same point, since the matrix M has rank 3 even in the point of intersection. But all fibers of φ are linear, so W must contain a plane through the two lines, this is absurd. Thus W is mapped birationally to \mathbf{P}^3 by φ .

It follows that the restriction of φ to R is a contraction onto a surface. Clearly it must contract the lines transverse to the planes. From the matrix M we see that the map is defined by cubics on each plane, and it is birational, so the image has

degree at least 3. Therefore, the inverse map is defined by hyperplanes or quadrics. Since W has degree 7 the inverse map must be defined by quadrics, and in fact with one basepoint, i.e. W is a Del Pezzo 3-fold W_7 . \square

5. Non-normal Calabi-Yau 3-folds in \mathbf{P}^5

We shall show that W_t for $t = 5, 6, 7$ is bilinked on a quintic hypersurface to a non-normal Calabi-Yau 3-fold. W_t is assumed to be general in the sense of section 4, including in the case of W_7 any Del Pezzo 3-fold characterized as in Proposition 4.10. Most of the argument goes through also for W_8 , but since we shall not need this later we do not conclude in this case (see Remark 5.3).

Lemma 5.1. *The general Del Pezzo 3-fold W_t for $t = 5, 6, 7, 8$ is contained in an irreducible quintic hypersurface. The general such quintic hypersurface is normal, it has double points along the singular lines of W_t and has only canonical singularities.*

Proof. We start with a general W_t and a general quintic hypersurface Q through W_t . Since W_t has non-normal double points along the singular lines and quintics generate its ideal, Q has multiplicity 2 at a general point on a singular line. Let

$$p' : \text{Bl}_s(\mathbf{P}^5) \rightarrow \mathbf{P}^5$$

be the blowup of \mathbf{P}^5 along the singular lines of W_t , and denote by W'_t and Q' the strict transforms of W_t and Q . Then W'_t has at most ordinary double points as singularities.

Next we consider the blowup

$$p'' : \text{Bl}_{s,W}(\mathbf{P}^5) \rightarrow \text{Bl}_s(\mathbf{P}^5)$$

of $\text{Bl}_s(\mathbf{P}^5)$ along W'_t . Over the singular points of W'_t , the blowup $\text{Bl}_{s,W}(\mathbf{P}^5)$ will have isolated double points. Denote by Q'' the strict transform of Q' and let W''_t be the strict transform of W'_t in Q'' . By the fundamental property of blowup, W''_t is a Cartier divisor on Q'' .

We now analyse the situation, assuming that we have chosen Q general. For $t \leq 7$, the ideal of W_t is generated by quintics by Proposition 4.5, so in this case Q'' is smooth outside W''_t . We claim that Q'' is smooth and that p'' restricted to Q'' is a small resolution of Q' . Since the quintics generate the ideal of W_t , the conormal sheaf of W'_t in $\text{Bl}_s(\mathbf{P}^5)$ twisted by the class of Q' is generated by its global sections. In particular, any divisor equivalent to Q' induces a section of this sheaf, which is a rank 2 bundle outside the singular points. Its zero locus is the subscheme of W'_t defined by the Jacobian ideal of Q' . For general Q this is a smooth curve which does not pass through the singular points of W'_t . Thus Q' is singular only along some smooth curve on W'_t not passing through the singularities of W'_t . The map p'' defines an isomorphism between W''_t and W'_t . Moreover since W''_t is a Cartier

divisor on Q'' it follows that Q'' is smooth and that p'' defines a small resolution of Q' showing that Q' has only canonical singularities. For W_8 one may show with [MAC] in the example of Lemma 4.9 that the baselocus of the quintics through W_8 is precisely the singular lines and the two planes. Therefore also in this case Q'' is smooth and we can argue as before.

Finally Q is smooth in codimension 1 and since it is a hypersurface, it is normal. It remains to prove that Q has canonical singularities along the singular lines. First recall that Q has multiplicity 2 at the generic point of the singular lines. Moreover the blowup p' defines a resolution of Q over the points on the singular lines outside the singular curve of Q' . Hence Q has transversal ordinary double points outside the finitely many points on the singular lines which are the intersection with the image under p' of the singular curve of Q' . In particular Q has canonical singularities outside these finitely many points. Combining this with the argument that p'' defines a small resolution of Q' gives the claim of the lemma. \square

Proposition 5.2. *The general Del Pezzo 3-fold W_t for $t = 5, 6, 7$ is bilinked $(5, 4)$ and $(5, 5)$ on a quintic hypersurface to a variety Y of degree $t + 5$ which has non-normal singularities along the singular lines of W_t . The normalization of Y is a smooth Calabi-Yau 3-fold.*

Proof. We start with a general quintic hypersurface Q which contains W_t as in Lemma 5.1, and keep the same setup and notation as in the above proof. In addition we let $H_{Q'}$ and $H_{Q''}$ be the pullback by p' and p'' of a hyperplane H restricted to Q . A 3-fold Y bilinked in hypersurfaces of degree 4 and 5 to W_t on Q is nothing but a Weil divisor equivalent to $W_t + H_Q$, where H_Q is the restriction of H to Q . The strict transform Y'' of Y on Q'' is a Cartier divisor linearly equivalent to $W_t'' + H_{Q''}$. To analyze the singularities and the canonical sheaf of Y we perform adjunction on the smooth 4-fold Q'' .

Let E'' be the pullback of the exceptional divisor of $\text{Bl}_s(\mathbf{P}^5)$ to $\text{Bl}_{s,W}(\mathbf{P}^5)$. Then the canonical line bundle on Q'' is $\mathcal{O}_{Q''}(E'' - H_{Q''})$. By adjunction on Q'' , the canonical line bundle on W_t'' is $\mathcal{O}_{W_t''}(W_t'' + E'' - H_{Q''})$. Consider now the normalization \tilde{W}_t of W_t . The map $W_t'' \rightarrow W_t$ factors through this normalization. On \tilde{W}_t there are no exceptional divisors since the singular locus has codimension 2. Therefore, the map $W_t'' \rightarrow \tilde{W}_t$ is the blowup along a curve with exceptional line bundle $\mathcal{O}_{W_t''}(E'')$. But \tilde{W}_t is a Del Pezzo 3-fold with only canonical Gorenstein singularities, so the canonical divisor of W_t'' is also $\mathcal{O}_{W_t''}(E'' - 2H_{Q''})$. Therefore $\mathcal{O}_{W_t''}(W_t'' - H_{Q''}) = \mathcal{O}_{W_t''}(-2H_{Q''})$ and we obtain the equality

$$\mathcal{O}_{W_t''}(W_t'' + H_{Q''}) = \mathcal{O}_{W_t''}.$$

Now, consider the exact sequences of sheaves on Q''

$$0 \longrightarrow \mathcal{O}_{Q''}(H_{Q''}) \longrightarrow \mathcal{O}_{Q''}(W_t'' + H_{Q''}) \longrightarrow \mathcal{O}_{W_t''}(W_t'' + H_{Q''}) \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{O}_{Q''} \longrightarrow \mathcal{O}_{Q''}(H_{Q''}) \longrightarrow \mathcal{O}_{H_{Q''}}(H_{Q''}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{O}_{H_{Q''}} \longrightarrow \mathcal{O}_{H_{Q''}}(H_{Q''}) \longrightarrow \mathcal{O}_{H_{Q''} \cap H'_{Q''}}(H_{Q''}) \longrightarrow 0$$

for general hyperplanes H and H' . Since Q and $Q \cap H$ have only canonical singularities, Q'' and $H_{Q''}$ are regular. Thus the second and third sequence remain exact on global sections. Furthermore $H \cap H' \cap Q$ is a smooth quintic surface, so $h^1(\mathcal{O}_{H \cap H' \cap Q}(H)) = h^1(\mathcal{O}_{H_{Q''} \cap H'_{Q''}}(H_{Q''})) = 0$. Therefore $h^1(\mathcal{O}_{H_{Q''}}(H_{Q''})) = h^1(\mathcal{O}_{Q''}(H_{Q''})) = 0$ and also the first sequence remains exact after taking global sections. Thus the line bundle in the middle is generated by global sections, i.e. the linear system $|W_t'' + H_{Q''}|$ of divisors on Q'' has no basepoints. By Bertini, we may conclude that a general member Y'' of this linear system is smooth.

The exceptional locus of p'' on Q'' is a surface scroll, and W_t'' intersects this scroll in a section over the smooth base curve. The restriction of the linear system $|W_t'' + H_{Q''}|$ to this scroll has no basepoints so the general member is again a section over the base curve. Therefore the restriction of p'' to Y'' is also an isomorphism, and the image Y' on Q' is smooth, while Y is singular only along the singular lines. Since W_t has non-normal double points along the singular lines, the same is the case for Y .

The canonical line bundle on Y'' is easily computed by adjunction on Q'' . In fact it is

$$\mathcal{O}_{Y''}(Y'' + E'' - H_{Q''}) = \mathcal{O}_{Y''}(W_t'' + E'').$$

Since $\mathcal{O}_{W_t''}(Y'') = \mathcal{O}_{W_t''}(W_t'' + H_{Q''}) = \mathcal{O}_{W_t''}$, we have $\mathcal{O}_{Y''}(W_t'') = \mathcal{O}_{Y''}$. Therefore the canonical line bundle on Y'' is $\mathcal{O}_{Y''}(E'')$.

We turn to the singular variety Y . It is singular precisely along the singular lines of W_t where it has non-normal double points. Consider the normalization $\tilde{Y} \rightarrow Y$. As in the case of W_t , the resolution of singularities $Y'' \rightarrow Y$ factors through this normalization. We will show that \tilde{Y} is already smooth. For this we first specialize the quintic hypersurface Q to a hypersurface containing W_t and a 3-fold Z that is smooth along the singular lines of W_t . In case $t = 7$, we take Z to be the Segre cubic scroll R , while in the cases $t = 5$ and $t = 6$ we may take Z to be one and two \mathbf{P}^3 's respectively. We make the computation explicit in the case when $t = 7$, which we will use later. The other cases are similar. Assume that Q is a quintic which contains W_t and the rational cubic scroll R . Let $Q_0 \rightarrow Q$ be the blow up of Q along R . This defines a small resolution of the singularities of Q along the lines. Over each line there is an exceptional scroll. This scroll is isomorphic to some Hirzebruch surface F_a for some $a \geq 0$. The strict transform of W_t meets this scroll in a rational curve which is a bisection on the scroll, since W_t is double along the line. Since Y as a Weil divisor is equivalent to $W_t + H_Q$ on Q , and the strict transform of H_Q intersects the scroll in a ruling, we get that the strict transform of Y meets the exceptional scroll in a bisection which is an elliptic curve. The normalization of Y factors through this small resolution and, therefore, has elliptic curves lying over the singular lines. In particular the normalization is smooth over the singular lines. By deformation to the general quintic Q , the normalization of Y is smooth.

Now, the normalization \tilde{Y} of Y is isomorphic to Y in codimension 1, therefore the canonical line bundle $\mathcal{O}_{Y''}(E'')$ on Y'' is the exceptional line bundle of the map $Y'' \rightarrow \tilde{Y}$ and the canonical line bundle of \tilde{Y} is trivial.

The irregularity of \tilde{Y} equals the irregularity of Y'' , since they are birational and both are smooth. But on Y'' we get

$$h^1(\mathcal{O}_{Y''}) = h^2(\mathcal{O}_{Q''}(-W_t'' - H_{Q''})) = h^1(\mathcal{O}_{W_t''}(-H_{Q''})) = 0$$

so the normalization of Y is regular. Thus the normalization of Y is a smooth Calabi-Yau 3-fold and the proposition follows. \square

Remark 5.3. The above proof goes through also for W_8 , except for the question of a smooth normalization.

6. Equations of elliptic scrolls in \mathbf{P}^5

In the remaining sections we prove that the union of two elliptic scrolls which intersect along an abelian surface is bilinked to a Del Pezzo 3-fold W_7 . On the way we describe the family of these reducible Calabi-Yau 3-folds using Heisenberg symmetry. But first we study the ideal of elliptic scrolls without using this symmetry.

Recall from section 3 that two elliptic scrolls whose intersection is a $(1,6)$ -polarized abelian surface in \mathbf{P}^5 , are each singular along three lines. Therefore we restrict our attention to this kind of elliptic scrolls. We start with a lemma which gives a quick construction of such scrolls.

Proposition 6.1. *The union of three \mathbf{P}^3 's in \mathbf{P}^5 , which meet pairwise in lines, is linked $(3,3)$ to an elliptic scroll. Furthermore any elliptic 3-fold scroll of degree 6, singular along three lines which span \mathbf{P}^5 , is linked $(3,3)$ to three \mathbf{P}^3 's.*

Proof. First, consider three \mathbf{P}^3 's which meet pairwise in lines and two general cubic hypersurfaces containing them. The linked variety X has degree 6 and sectional genus 1 (cf. [PS]). Since the complete intersection has trivial canonical bundle, each component intersects the rest along an anticanonical divisor. Therefore X meets each of the \mathbf{P}^3 's in a quartic surface, singular along the two lines of intersection with the other two \mathbf{P}^3 's. These quartic surfaces clearly are elliptic scrolls: Through every point on the surface outside the two singular lines there is a unique line in the surface through the point intersecting the two lines, by Bezout, so the quartic surface is a scroll of lines. Through every point on the singular lines there are two rulings of the scroll. In the Grassmannian of lines the curve parametrizing the rulings has a double cover to each of the two singular lines, so the curve is of type $(2,2)$ on $\mathbf{P}^1 \times \mathbf{P}^1$, i.e. it is elliptic. The normalization of X therefore contains elliptic scrolls in parts of hyperplane sections. On the other hand, residual to each quartic surface in hyperplane sections, there is a pencil of surfaces of degree two on

X . Residual to the \mathbf{P}^3 they form part of a complete intersection of two quadrics in the hyperplane. The other two \mathbf{P}^3 's intersect this hyperplane in two planes passing through a point, so the surface of degree two on X must be two planes residual to these in the complete intersection. This displays the scroll structure of X . Clearly the scroll is elliptic.

Next, consider an elliptic scroll X_F of degree 6 and singular along three lines that span \mathbf{P}^5 . Let

$$\varphi : \mathbf{P}(\mathcal{E}) \rightarrow X_F$$

be the normalization map. Like in section 3 the three lines correspond to a decomposition

$$\mathcal{E} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2,$$

where the \mathcal{L}_i are line bundles of degree 2. We have

$$H^0(F, \mathcal{E}) \cong H^0(X_F, \mathcal{O}_{X_F}(1)) \cong H^0(\mathbf{P}^5, \mathcal{O}_{\mathbf{P}^5}(1))$$

since the linear system that maps $\mathbf{P}(\mathcal{E})$ into \mathbf{P}^5 is complete. There is an isomorphism

$$H^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)) \cong H^0(F, S^n \mathcal{E}) \cong H^0(F, \mathcal{L}_0^n \oplus \mathcal{L}_0^{n-1} \otimes \mathcal{L}_1 \oplus \dots \oplus \mathcal{L}_2^n)$$

where the number of summands is

$$\binom{n+2}{2} = \frac{(n+1)(n+2)}{2}$$

and the degree of each summand is $2n$. This shows that

$$h^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)) = h^0(F, S^n \mathcal{E}) = n(n+1)(n+2).$$

We now compare this to the situation on the singular scroll X_F . Recall that there are three sections $F_i \subset \mathbf{P}(\mathcal{E})$ such that the map φ restricted to F_i induces a double cover $\varphi : F_i \rightarrow L_i$ onto a line L_i . This is branched over 4 points and hence

$$\varphi_* \mathcal{O}_{F_i} = \mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(-2),$$

and we have the exact sequence

$$0 \rightarrow \mathcal{O}_{L_i} \rightarrow \varphi_* \varphi^* \mathcal{O}_{L_i} \rightarrow \mathcal{O}_{L_i}(-2) \rightarrow 0$$

where $\varphi_* \varphi^* \mathcal{O}_{L_i} = \varphi_* \mathcal{O}_{F_i}$.

Lemma 6.2. *There is an exact sequence of sheaves*

$$0 \rightarrow \mathcal{O}_{X_F} \rightarrow \varphi_* \mathcal{O}_{\mathbf{P}(\mathcal{E})} \rightarrow \bigoplus_{i=0}^2 \mathcal{O}_{L_i}(-2) \rightarrow 0.$$

Proof. The cokernel clearly has support on the locus on X_F where the map is not an isomorphism, i.e. on the three lines L_i . Restricted to these lines the exact sequence reduces to the one above. Since the lines are disjoint, the cokernel is a direct sum. \square

Lemma 6.3. *After tensoring with $\mathcal{O}_{X_F}(n)$ for any nonnegative n the sequence of Lemma 6.2 is exact on global sections.*

Proof. For $n = 0, 1$ the statement is immediate, since the third term has no sections. Furthermore $h^1(\mathcal{O}_{X_F}(1)) = h^1(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)) = 0$ and $h^2(\mathcal{O}_{X_F}(n)) = h^2(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(n)) = 0$ for $n \geq 1$. Let H be a general hyperplane, and $P = \mathbf{P}^3$ be a general 3-space inside H . Then $X_F \cap P$ is a smooth elliptic curve, so $h^1(\mathcal{O}_{X_F \cap P}(n)) = 0$ for $n \geq 1$. Furthermore $h^1(\mathcal{O}_{X_F \cap H}(1)) = 0$. Inductively $h^1(\mathcal{O}_{X_F \cap H}(n)) \leq h^1(\mathcal{O}_{X_F \cap H}(n-1)) = 0$ for $n \geq 2$, and similarly

$$h^1(\mathcal{O}_{X_F}(n)) \leq h^1(\mathcal{O}_{X_F}(n-1)) = 0$$

for $n \geq 2$ and the lemma follows. \square

Lemma 6.3 allows us to compute

$$h^0(\mathcal{O}_{X_F}(n)) = n(n+1)(n+2) - 3(n-1), \quad n \geq 1.$$

The natural maps $H^0(\mathcal{O}_{\mathbf{P}^5}(n)) \rightarrow H^0(\mathcal{O}_{X_F}(n))$ and the dimension

$$h^0(\mathcal{O}_{\mathbf{P}^5}(n)) = \dim S^n H^0(\mathcal{E}) = \binom{n+5}{5},$$

gives us

$$h^0(\mathcal{I}_{X_F}(n)) - h^1(\mathcal{I}_{X_F}(n)) = \binom{n+5}{5} + 3(n-1) - n(n+1)(n+2).$$

In particular $h^0(\mathcal{I}_{X_F}(3)) \geq 2$. Clearly the complete intersection of two cubics through X_F contains the \mathbf{P}^3 's spanned by the pairs of singular lines, therefore $h^0(\mathcal{I}_{X_F}(3)) = 2$ and Proposition 6.1 follows. \square

Corollary 6.4. $h^0(\mathcal{I}_{X_F}(5)) = 54$.

Proof.

$$h^0(\mathcal{I}_{X_F}(5)) - h^1(\mathcal{I}_{X_F}(5)) = 54,$$

as follows from the computation in the proof of Proposition 6.1. But $X_F \cap \mathbf{P}^2$ for a general \mathbf{P}^2 is 4-regular in the sense of Castelnuovo-Mumford, so $h^1(\mathcal{I}_{X_F}(n)) = 0$ for $n \geq 3$. \square

7. Heisenberg symmetry of elliptic scrolls in \mathbf{P}^5

First we collect some basic observations. Recall the Heisenberg group of level 6:

$$1 \rightarrow \mu_6 \rightarrow H_6 \rightarrow \mathbf{Z}/6 \times \mathbf{Z}/6 \rightarrow 0.$$

Given any subgroup $G \subset \mathbf{Z}/6 \times \mathbf{Z}/6$ we can consider the preimage

$$H(G) := \pi^{-1}(G) \subset H_6.$$

Let $\langle x_0, \dots, x_5 \rangle$ be a basis for a 6-dimensional complex vector space V and let $\langle e_0, \dots, e_5 \rangle$ be a basis for V^* . The Schrödinger representation $\rho : H \rightarrow GL(V, \mathbf{C})$ is defined by

$$\sigma : x_i \mapsto x_{i+1}, \quad \tau : x_i \mapsto \rho^i x_i$$

where $\rho = e^{2\pi i/6}$ and indices are taken modulo 6. It defines, by restriction, representations

$$\rho_G : H(G) \rightarrow GL(V, \mathbf{C}).$$

The group $\mathbf{Z}/6 \times \mathbf{Z}/6$ contains unique subgroups isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/2$, resp. $\mathbf{Z}/3 \times \mathbf{Z}/3$. The preimages of these groups in H_6 are isomorphic to the Heisenberg groups H_2 and H_3 of level 2 and 3. Note, however, that in the case of H_3 the induced representation differs from the Schrödinger representation by the non-trivial automorphism of the Galois group.

Lemma 7.1. (i) *There are 4 subgroups isomorphic to $\mathbf{Z}/3$ in $\mathbf{Z}/3 \times \mathbf{Z}/3$.*

(ii) *There are 4 subgroups isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/6$ in $\mathbf{Z}/6 \times \mathbf{Z}/6$.*

Proof. Claim (i) is trivial. Every subgroup isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/6$ is generated by the elements of order 2 and one element of order 3 in $\mathbf{Z}/6 \times \mathbf{Z}/6$, i.e. by the subgroup $\mathbf{Z}/2 \times \mathbf{Z}/2 \subset \mathbf{Z}/6 \times \mathbf{Z}/6$ and a subgroup of order 3 of the group $\mathbf{Z}/3 \times \mathbf{Z}/3 \subset \mathbf{Z}/6 \times \mathbf{Z}/6$. This shows (ii). \square

Remark 7.2. At the same time this gives us a natural 1 : 1 correspondence between the 4 groups of part (i) of Lemma 7.1 and the 4 groups of part (ii). We shall use this frequently in what follows.

We shall denote the 4 subgroups of $\mathbf{Z}/6 \times \mathbf{Z}/6$ which are isomorphic to $\mathbf{Z}/2 \times \mathbf{Z}/6$ by K_1, \dots, K_4 . For every group G in $\mathbf{Z}/6 \times \mathbf{Z}/6$ we set

$$H(G)_\iota = \langle H(G), \iota \rangle \subset G_6.$$

We denote by G_2 resp. G_3 the groups $\langle H_2, \iota \rangle$, resp. $G_3 = \langle H_3, \iota \rangle$. Next we need some elementary representation theory. We denote by U the Schrödinger representation of H_2 .

Lemma 7.3. (i) *As an H_2 -module $V \cong 3U$,*

(ii) *as a G_2 -module $V \cong 2U_+ \oplus U_-$, where U_+ , resp. U_- means that ι acts by +1, resp. -1 on U ,*

(iii) *as K_i -modules $V \cong U \oplus U' \oplus \bar{U}'$. Here the subgroup of K_i which is isomorphic to $\mathbf{Z}/3$, acts on U' by a non-trivial character and on \bar{U}' , by the inverse of this character. The involution ι leaves U fixed and interchanges U' and \bar{U}' .*

Proof. (i), (ii) We can decompose V as an H_2 -module as follows:

$$\langle x_0, x_3 \rangle, \langle x_2, x_5 \rangle, \langle x_4, x_1 \rangle.$$

From this the claim is obvious.

(iii) It is enough to consider one of the groups K_i . The others can be done in the same way, or one can use the normalizer N_6 of H_6 in $GL(6, \mathbf{C})$. Here we shall consider the subgroup K given by τ^2 . One immediately checks that τ^2 acts by $1, \rho^4, \rho^2$ ($\rho = e^{2\pi i/6}$) on the above submodules of V , hence giving the claim. \square

We next want to associate basic geometric objects in \mathbf{P}^5 to the subgroups $K_i, i = 1, 2, 3, 4$.

Lemma 7.4. (i) *To every subgroup $H(K_i), i = 1, 2, 3, 4$ of H_6 one can associate a unique set of 3 lines in \mathbf{P}^5 which is an H_6 -orbit such that $H(K_i)$ is the stabilizer of each of these lines. The distinguished subgroup of order 3 in K_i fixes these lines pointwise.*

(ii) *Every H_6 -orbit of lines in \mathbf{P}^5 consisting of 3 lines is one of the above.*

Proof. Let $\{L_1, L_2, L_3\}$ be an H_6 -orbit of lines in \mathbf{P}^5 . Then every line L_j has a stabilizer in $\mathbf{Z}/6 \times \mathbf{Z}/6$ of order 12. This must then be one of the groups $K_i, i = 1, 2, 3, 4$. Without loss of generality it suffices to consider the group generated by $\langle \tau, \sigma^3 \rangle$. Then the action of τ on the vector space associated to such a line L_j splits into a sum of two different characters. Therefore L_j is spanned by two basis vectors e_k, e_l . To obtain invariance under σ^3 the only possibilities are $\langle e_0, e_3 \rangle, \langle e_1, e_4 \rangle$ and $\langle e_2, e_5 \rangle$. Furthermore the distinguished subgroup order 3 generated by τ^2 fixes these three lines pointwise. \square

Lemma 7.5. (i) *To every subgroup $H(K_i), i = 1, 2, 3, 4$ of H_6 one can associate a unique set of three 3-spaces in \mathbf{P}^5 which is an H_6 -orbit such that $H(K_i)$ is the stabilizer of each of these $\mathbf{P}^{3'}$ s.*

(ii) *Every H_6 -orbit of 3-spaces in \mathbf{P}^5 consisting of three $\mathbf{P}^{3'}$ s is one of the above.*

Proof. This is the dual statement to Lemma 7.4. Given three lines L_1, L_2, L_3 which form an H_6 -orbit, the three $\mathbf{P}^{3'}$ s are the spaces spanned by two of these lines. \square

Next we turn to the space of cubic forms $H^0(\mathcal{O}_{\mathbf{P}^5}(3))$ which we want to study as an H_6 -, resp. G_6 -module.

Lemma 7.6. *The G_6 -module $H^0(\mathcal{O}_{\mathbf{P}^5}(3))$ is a sum of four 2-dimensional and twelve 4-dimensional representations. As H_3 -representation it is a sum of characters. The trivial character corresponds to the four pencils. The other 8 come in pairs (given by the involution ι) and each pair determines three 4-dimensional irreducible G_6 -representations. The subspace of 2-dimensional representations is spanned by*

$$\begin{aligned} & \langle x_0^3 + x_2^3 + x_4^3, x_1^3 + x_3^3 + x_5^3 \rangle \\ & \langle x_0x_2x_4, x_1x_3x_5 \rangle \\ & \langle x_3^2x_0 + x_5^2x_2 + x_1^3x_4, x_4^2x_1 + x_6^2x_3 + x_2^3x_5 \rangle \\ & \langle x_1x_2x_3 + x_3x_4x_5 + x_5x_0x_1, x_2x_3x_4 + x_4x_5x_0 + x_0x_1x_2 \rangle. \end{aligned}$$

Proof. This is a straightforward computation. \square

Since all the 2-dimensional representations are mutually isomorphic this defines a \mathbf{P}^3 of pencils of cubics.

Proposition 7.7. *For every G_6 -orbit of 3 lines in \mathbf{P}^5 , there is a unique pencil of G_6 -invariant pencils of cubics containing these lines.*

Proof. We can assume that the 3 lines in question are $\langle e_0, e_3 \rangle$, $\langle e_1, e_4 \rangle$ and $\langle e_2, e_5 \rangle$. A general pencil of G_6 -invariant pencils of cubics is of the form

$$\begin{aligned} & a\langle x_0^3 + x_2^3 + x_4^3, x_1^3 + x_3^3 + x_5^3 \rangle \\ & \quad + b\langle x_0x_2x_4, x_1x_3x_5 \rangle \\ & + c\langle x_3^2x_0 + x_5^2x_2 + x_1^2x_4, x_4^2x_1 + x_0^2x_3 + x_2^2x_5 \rangle \\ & + d\langle x_1x_2x_3 + x_3x_4x_5 + x_5x_0x_1, x_2x_3x_4 + x_4x_5x_0 + x_0x_1x_2 \rangle. \end{aligned}$$

Such a pencil contains the above lines if and only if $a = c = 0$. \square

Remark 7.8. (i) It is also easy to determine the pencil of pencils containing the other minimal H_6 -orbits of lines. E.g. the three lines fixed by the elements σ and τ^3 are the line $\langle (1, 1, 1, 1, 1, 1), (1, -1, 1, -1, 1, -1) \rangle$ and its τ -translates. The corresponding pencil of pencils is given by $3a + b = c + d = 0$.

(ii) Every pencil of G_6 -invariant pencils of cubics containing a minimal orbit $\{L_1, L_2, L_3\}$ also contains the three $\mathbf{P}^{3'}$ s spanned by two of these lines. (This can be seen by direct inspection). Hence every such pencil has a base locus consisting of three $\mathbf{P}^{3'}$ s and a residual 3-fold X of degree 6. By Proposition 6.1, X is an elliptic scroll. Furthermore, every elliptic 3-fold scroll X_F , as in section 3, is G_6 -invariant, so it is contained in a G_6 -invariant pencil of cubics. Hence for a general pencil of cubics the residual X must be of the form X_E for a suitable elliptic curve E .

Our next aim is to study the Heisenberg action on the embedded abelian surfaces $E \times F$. Recall that every abelian surface with a very ample $(1, 6)$ -polarization can be embedded G_6 -equivariantly into \mathbf{P}^5 and that the choice of such an embedding is equivalent to the choice of a level-6 structure on $E \times F$ (i.e. a canonical level structure associated to the polarization H by which we mean a symplectic basis of the kernel of the map $\lambda_H : A \rightarrow \hat{A}$). We denote the family of all G_6 -invariant abelian surfaces in \mathbf{P}^5 with two plane elliptic fibrations by \mathcal{A} . As in section 1 we start with curves E and F and a $3 : 1$ morphism $\gamma : E \rightarrow F$. We can assume that $E = \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$, $F = \mathbf{C}/(\mathbf{Z}3\tau + \mathbf{Z})$ and that the map γ is induced by $\gamma(z) = 3z$. We denote the generators τ and 1 of the lattice $\mathbf{Z}\tau + \mathbf{Z}$ by e_1 and e_3 and set $s_6 = e_1/6$ and $t_6 = e_3/6$. Moreover we denote the generators 3τ and 1 of the lattice $\mathbf{Z}3\tau + \mathbf{Z}$ by e_2 and e_4 . Then $\gamma(e_1) = e_2, \gamma(e_3) = 3e_4$. The point $u_6 = \gamma(s_6)$ is represented by $e_2/6$ and $\gamma(t_6) = 3v_6$ is represented by $e_4/2$. We choose v_6 as the point represented by $e_4/6$. In this set-up the product $A = E \times F$ is given by the

period matrix

$$\begin{pmatrix} \tau & 0 & 1 & 0 \\ 0 & 3\tau & 0 & 1 \end{pmatrix}.$$

The first step is to understand the polarization $H = 2E + \Gamma$ in terms of a Riemann form. This is necessary to understand the level-6 structures on A . First we look at the semi-positive line bundle defined by E . The corresponding Riemann form with respect to the lattice $L = \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$ is clearly given by

$$H_E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

Next we want to identify the form H_Γ with respect to the chosen basis e_1, \dots, e_4 . Since $\Gamma \cdot F = 1$ we can write $A = \Gamma \times F$. By abuse of notation, let $\gamma = (\text{id}, \gamma) : E \rightarrow E \times F$ by the embedding of E into A . Then $\gamma(E) = \Gamma$. We have $\gamma(e_1) = e_1 + e_2 =: f_1$, $\gamma(e_3) = e_3 + 3e_4 =: f_2$. We can also choose f_1, f_2, e_2, e_4 as a basis for L . With respect to this basis the semi-positive form H_Γ is given by $(e_2, e_4) = 1$ and all other products 0. A straightforward calculation then shows that in terms of the basis e_1, \dots, e_4 the form H_Γ is given by:

$$H_\Gamma = \begin{pmatrix} 0 & 0 & 3 & -1 \\ 0 & 0 & -3 & 1 \\ -3 & 3 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{pmatrix}.$$

Since $H = 2E + \Gamma$ it follows that the corresponding form with respect to the basis e_1, \dots, e_4 is given by

$$H = \begin{pmatrix} 0 & 0 & 3 & -1 \\ 0 & 0 & -3 & 3 \\ -3 & 3 & 0 & 0 \\ 1 & -3 & 0 & 0 \end{pmatrix}.$$

Note that this is indeed the form associated to a $(1, 6)$ -polarization, since

$$\det \begin{pmatrix} 3 & -1 \\ -3 & 3 \end{pmatrix} = 6.$$

Our next aim is to identify the group $\Theta(H) = \ker(\lambda_H : A \rightarrow \hat{A})$ as a subgroup of $A^{(6)} = E^{(6)} \times F^{(6)}$. What we have to do is to find a basis of L^\vee/L where L^\vee is the dual lattice with respect to the form H . General theory tells us that $L^\vee/L \cong \mathbf{Z}/6 \times \mathbf{Z}/6$. It is a straightforward calculation to check that $e_1/2 + e_2/6, e_3/6 + e_4/2 \in L$. We can take these elements as generators of L^\vee/L . Note that as points in $A = E \times F$ these are just the points $(3s_6, u_6)$ and $(t_6, 3v_6)$.

Let $w_1 = e_1/2 + e_2/6, w_2 = e_3/6 + e_4/2$. Then a straightforward calculation shows that for the Weil pairing with respect to H we have

$$(w_1, w_2) = \left(e^{2\pi i/6} \right),$$

i.e. these points define a level-6 structure. Note that $2w_1 \in F = \{0\} \times F$ and $2w_2 \in E = E \times \{0\}$. In particular the 2 groups of order 3 generated by $2w_1$ and $2w_2$ each respect one of the two plane elliptic fibrations of $A = E \times F$. Since the embedded abelian surface $A = E \times F$ is G_6 -invariant the same holds for the scrolls X_E and X_F defined by the 2 plane cubic fibrations. Since the groups of order 3 generated by $2w_1$, resp. $2w_2$ each respect one of these fibrations it follows that they act trivially on the 3 singular lines of the scroll X_E , resp. X_F . In particular this gives 2 groups $H(K_i)$ and $H(K_j)$ which each has the 3 singular lines of the scrolls X_E , resp. X_F as one of its orbits. Any other choice of a level-6 structure on A gives an analogous picture. We shall return to this in a moment.

We now want to understand the variety \mathcal{A} parametrizing G_6 -invariant abelian surfaces with 2 plane cubic fibrations. We have already observed that each of these two fibrations determines a singular scroll X_E , resp. X_F and a group $H(K_i)$, resp. $H(K_j)$. The abelian surface A is the intersection of the scrolls X_E and X_F (cf. Proposition 3.5.) This defines a decomposition of the family \mathcal{A} into six families \mathcal{A}_{ij} where $\{i, j\} \subset \{1, 2, 3, 4\}$. We want to exhibit a concrete parametrization of the families \mathcal{A}_{ij} , thereby also showing that the \mathcal{A}_{ij} form six irreducible components. To do this we go back to an elliptic curve E as before and the level-6 structure on E given by (s_6, t_6) . We can perform the above construction and associate to these data the surface $A = E \times F$, the polarization $H = 2E + \Gamma$ and the level-6 structure (w_1, w_2) . This gives us a morphism

$$\psi_{ij} : X^0(6) \rightarrow \mathcal{A}_{ij}$$

from the (open) elliptic modular curve $X^0(6)$ parametrizing elliptic curves with a level-6 structure to \mathcal{A}_{ij} . Note that the (compact) modular curve $X(6)$ is an elliptic curve.

Lemma 7.9. *The map $\psi_{ij} : X^0(6) \rightarrow \mathcal{A}_{ij}$ is surjective onto the component \mathcal{A}_{ij} and has degree 3.*

Proof. Here we shall treat the case where K_i and K_j are the groups determined by $\langle \sigma^2 \rangle$ and $\langle \tau^2 \rangle$. This is no loss of generality. Going back to the abelian surface $A = E \times F$ we want to study the possible embeddings of A into \mathbf{P}^5 such that $A \in \mathcal{A}_{ij}$. Since G_6 -invariant embeddings of A correspond to the choice of a level-6 structure (w'_1, w'_2) we have to look for those level-6 structures (w'_1, w'_2) such that $2w'_1 \in F = \{0\} \times F$ and $2w'_2 \in E = E \times \{0\}$ or $2w'_1 \in E$ and $2w'_2 \in F$. These two cases correspond to changing the role of E and F and it is, therefore, enough to look at the first possibility, namely $2w'_1 \in F$ and $2w'_2 \in E$. Since (w_1, w_2) is a basis of $\Theta(H)$ we can write $w'_1 = \alpha w_1 + \beta w_2$, $w'_2 = \gamma w_1 + \delta w_2$. Moreover $2w_1 \in F$ and hence $2w'_1 \in F$ if and only if $2\beta w_2 \in F$ which is only the case for $\beta = 0$ or 3. Moreover w'_1 has to be an element of order 6. This gives us three possibilities for $\pm w'_1$, namely $w_1, w_1 + 3w_2, 2w_1 + 3w_2$. Since (w'_1, w'_2) and $(-w'_1, -w'_2)$ define the same level-6 structure we can assume that w'_1 is one of the 3 points above. A similar argument can be applied to w'_2 and altogether we find the following 6

possibilities for level-6 structure (w'_1, w'_2) with $2w'_1 \in F$ and $2w'_2 \in E$:

$$(w_1, w_2), (w_1, w_2 + 3w_1), (w_1 + 3w_2, w_2), \\ (-2w_1 + 3w_2, w_2 + 3w_1), (-2w_1 + 3w_2, -2w_2 + 3w_1), (w_1 + 3w_2, -2w_2 + 3w_1).$$

For every pair (w'_1, w'_2) as above the pair $(-w'_2, w'_1)$ is a level-6 structure with $-2w'_2 \in E, 2w'_1 \in F$. In this way we obtain 12 level-6 structures which belong to the pair $\{i, j\}$. This fits in with the number of level-6 structures which is given by

$$\frac{1}{2}6^3 \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) = 72 = 12 \cdot 6,$$

where 6 corresponds to the number of pairs $\{i, j\} \subset \{1, 2, 3, 4\}$.

The map $\psi_{ij} : X^0(6) \rightarrow \mathcal{A}_{ij}$ was defined by associating to an elliptic curve E with level-6 structure (s_6, t_6) the abelian surface $A = E \times F$ with level-6 structure (w_1, w_2) . We want to prove that this map is surjective which implies in particular that \mathcal{A}_{ij} is irreducible. So we have to show that we can obtain all abelian surfaces $A = E \times F$ and all level-6 structures (w'_1, w'_2) as above by varying E and the level-6 structure (s_6, t_6) . Let (s'_6, t'_6) be any level-6 structure on E . To this we associate a surface $A' = E \times F'$ with $F' = E/\langle 2t'_6 \rangle$. If we want that $F' = F$ we must (at least for general F) have that $\langle 2t'_6 \rangle = \langle 2t_6 \rangle$. Moreover t'_6 must be a point of order 6. Up to sign this leaves us with the possibilities $t'_6 = t_6, 3s_6 + t_6, 3s_6 + 2t_6$. Altogether we obtain 18 possible level-6 structures, namely

$$(s_6 + 2it_6, t_6), (s_6 + (2i + 1)t_6, t_6) \\ (s_6 + 2it_6, 3s_6 + t_6), (-2s_6 + (2i + 1)t_6, 3s_6 + t_6) \\ (-2s_6 + (2i + 1)t_6, 3s_6 - 2t_6), (s_6 + (2i + 1)t_6, 3s_6 - 2t_6)$$

where in each case $i = 0, 1, 2$. Each of the values $i = 0, 1, 2$ gives the same level-6 structure on A . Hence under the morphism $\psi_{ij} : X^0(6) \rightarrow \mathcal{A}_{ij}$ the pairs $(E, (s'_6, t'_6))$ are mapped 3 : 1 to $(A = E \times F, (w'_1, w'_2))$ where (w'_1, w'_2) runs through all 6 possible level-6 structures on A with $2w'_1 \in F$ and $2w'_2 \in E$. In particular the map $X^0(6) \rightarrow \mathcal{A}_{ij}$ is 3 : 1 and surjective and \mathcal{A}_{ij} is irreducible. \square

Next we consider the family \mathcal{V} of G_6 -invariant scrolls which arise as \mathbf{P}^2 -scrolls defined by a plane cubic fibration of a G_6 -invariant abelian surface A . Such a scroll is singular along 3 lines which form an orbit of one of the groups $H(K_i)$ and hence there is a natural decomposition $\mathcal{V} = \mathcal{V}^1 \cup \mathcal{V}^2 \cup \mathcal{V}^3 \cup \mathcal{V}^4$ where \mathcal{V}^i is the set of those scrolls which are invariant under $H(K_i)$. These scrolls are in 1 : 1 correspondence with points in an open set of the pencil of pencils of cubics associated to the group $H(K_i)$ (cf. Remark 7.8. ii). In particular the varieties \mathcal{V}^i are irreducible and rational.

The G_6 -action on the elliptic scrolls restricts to pencils of G_6 -invariant abelian surfaces. We describe these before we return to the surfaces which lie on two scrolls.

Recall that the normalization $\varphi : \mathbf{P}_F(\mathcal{E}) \rightarrow X_F \subset \mathbf{P}^5$ of the scroll X_F is a \mathbf{P}^2 -bundle over the elliptic curve F associated to the rank 3 vector bundle

$$\mathcal{E} = \mathcal{L}_0 \oplus h^* \mathcal{L}_0 \oplus (h^2)^* \mathcal{L}_0, \quad (h^3 = \text{id}).$$

X_F is singular along 3 lines L_i , and

$$(h^i)^* \mathcal{L}_0 = \varphi^* \mathcal{O}_{L_i}(1), \quad i = 0, 1, 2$$

are line bundles of degree 2 on F . If we assume that say

$$L_0 = \langle e_0, e_3 \rangle, \quad L_1 = \langle e_1, e_4 \rangle, \quad \text{and} \quad L_2 = \langle e_2, e_5 \rangle,$$

then

$$\text{Stab}(L_i) = \langle \tau, \sigma^3 \rangle.$$

Recall from section 3 that we may find $\mathbf{P}_F(\mathcal{E})$ as a quotient of $\mathbf{P}^2 \times E$ by a subgroup $\mathbf{Z}/3 \subset H_3$. This subgroup leaves three sections $E \rightarrow \mathbf{P}^2 \times E$ invariant, and these three sections are mapped in the quotient to the three sections $F \rightarrow \mathbf{P}_F(\mathcal{E})$ which again are mapped 2 : 1 to the singular lines in X_F . In this set-up we may describe the G_6 -action on the abelian surfaces on X_F . These surfaces are all pulled back to anticanonical divisors on $\mathbf{P}_F(\mathcal{E})$. Recall that $h^0(\mathcal{O}_{\mathbf{P}(\mathcal{E})}(-K)) = 4$. Elements in $|-K|$ pull back to products

$$E' \times E \subset \mathbf{P}^2 \times E$$

where E' is a cubic curve invariant under the subgroup of order 3; in suitable coordinates it is defined by a form in the web

$$\langle x_0^3, x_1^3, x_2^3, x_0 x_1 x_2 \rangle.$$

These forms are precisely the invariants of degree 3 of a subgroup of order 3 in H_3 . In fact the elements of order 3 of H_6 which fix the lines L_i pointwise, leave each plane in X_F invariant and the lift to an action on $\mathbf{P}_F(\mathcal{E})$ and $\mathbf{P}^2 \times E$ which leaves each plane invariant and fixes the three special sections pointwise. These sections, in suitable coordinates, meet each plane in the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and the action is given by $\tau \in H_3$ in the plane.

The involution ι leaves L_0 fixed while L_1 and L_2 are interchanged. This also lifts to $\mathbf{P}^2 \times E$. The ι -invariant plane cubics in the above web form the net

$$\langle x_0^3, x_1^3 + x_2^3, x_0 x_1 x_2 \rangle.$$

G_6 finally permutes the three lines cyclically, so the corresponding plane cubics are defined by invariant forms in the variables permuted cyclically. This action is the one defined by $\sigma \in H_3$. The G_6 -invariant anticanonical divisors therefore correspond precisely to the Hesse-pencil

$$\langle x_0^3 + x_1^3 + x_2^3, x_0 x_1 x_2 \rangle.$$

As an H_3 -module $H^0(\mathcal{O}_{\mathbf{P}^2}(3))$ splits as the Hesse-pencil plus 8 characters. Two of these give rise to anticanonical divisors on $\mathbf{P}_F(\mathcal{E})$. The remaining 6 characters of H_3 give rise to bielliptic surfaces in $|-K+T|$ where T is some torsion divisor.

At this point we can also understand the different G_6 -embeddings of a scroll X_F into \mathbf{P}^5 . For this we start with a G_6 -invariant scroll X_F . Its desingularisation is $\mathbf{P}_F(\mathcal{E})$. First note that the Heisenberg group H_2 acts on the planes of X_F and that this induces actions of H_2 on the base curve F of $\mathbf{P}_F(\mathcal{E})$ and on the 3 sections which are mapped to the singular lines of X_F . To define a non-degenerate map from $\mathbf{P}_F(\mathcal{E})$ to \mathbf{P}^5 is the same as defining an isomorphism from $H^0(\mathcal{O}_{\mathbf{P}_F(\mathcal{E})}(1)) = H^0(\mathcal{E})$ to V . The decomposition $\mathcal{E} = \mathcal{L}_0 \oplus h^*\mathcal{L}_0 \oplus (h^2)^*\mathcal{L}_0$ defines a decomposition $H^0(\mathcal{E}) = U_1 \oplus U_2 \oplus U_3$ where each of the spaces U_i is the space of sections of a degree 2 line bundle on F and hence has dimension 2. The level 2 structure on F which comes from the action of G_6 on X_F gives us an identification (unique up to a scalar) of each of the U_i with the H_2 -module U . Now we pick one of the 4 subgroups K_j . Recall that as an H_2 -module $V = 3U$, whereas as an $H(K_j)$ -module $V = U \oplus U' \oplus \bar{U}'$ (cf. Lemma 7.3). In order to map the 3 decomposing sections of $\mathbf{P}_F(\mathcal{E})$ to the singular lines associated to the group K_j we must map each of the spaces U_i to one of the spaces U, U' and \bar{U}' . The group G_6 acts transitively on the lines associated to K_j and hence we can (up to an element in G_6) assume that U_1, U_2, U_3 map to U, U', \bar{U}' . This defines the isomorphism from $H^0(\mathcal{E})$ to V up to an element in $(\mathbf{C}^*)^3$. On the other hand the Heisenberg group H_3 acts irreducibly on the 3-dimensional space given by the decomposition $V = 3U$. Hence, by Schur's lemma, the isomorphism from $H^0(\mathcal{E})$ to V is uniquely defined (up to a scalar). This shows that given a G_6 -invariant embedding of X_F in \mathbf{P}^5 we can find four such embeddings, one for each of the subgroups K_j .

We can now consider the incidence correspondence

$$\mathcal{I}(A, X) = \{(A, X) | A \in \mathcal{A}, X \in \mathcal{V}, A \subset X\} \subset \mathcal{A} \times \mathcal{V}.$$

Proposition 7.10. *This is a 2 : 3 correspondence.*

Proof. Clearly $A \in \mathcal{A}$ is the intersection of two scrolls. On the other hand, the number of such surfaces in a scroll is the answer to the question: how many abelian surfaces are there in the pencil $H \subset |-K_{\mathbf{P}_F(\mathcal{E})}|$ coming from the Hesse pencil which are isomorphic to the product $E \times F$? For this we want to find an embedding $E \subset \mathbf{P}^2 \times E$ which after projection to \mathbf{P}^2 maps E to an element in the Hesse pencil which is $\mathbf{Z}/3$ -equivariant (here $\mathbf{Z}/3$ is the group which acts on $\mathbf{P}^2 \times E$ with quotient $\mathbf{P}_F(\mathcal{E})$). To embed E as an element in the Hesse pencil is the same as choosing a level 3 structure on E . Say $\mathbf{Z}/3$ acts on E by translation with an element σ' of order 3. So we have to ask in how many ways we can extend σ' to a level 3 structure. If τ' is another 3-torsion point with $(\sigma', \tau') = 1$ (here $(\ , \)$ is the Weil pairing), we have the possibilities (σ', τ') , $(\sigma', \tau'\sigma')$, $(\sigma', \tau'\sigma'^2)$ and no others. This gives us the three possibilities. \square

Remark 7.11. Notice that the three choices of $\mathbf{Z}/3$ -subgroups of $\mathbf{Z}/3 \times \mathbf{Z}/3$ correspond precisely to the three subgroups K_i distinct from the subgroup K_j which stabilizes the three singular lines of the scroll X_F .

Corollary 7.12. *Given two distinct subgroups K_i and K_j , the incidence correspondence $\mathcal{I}(A, X)$ defines a $1 : 1$ correspondence between elliptic scrolls X whose singular lines are invariant under these two subgroups.*

Proof. A scroll singular along one of the triples of lines contains exactly one abelian surface which forms the intersection with a scroll singular along the other triple of lines. \square

Corollary 7.13. *The abelian surfaces with two plane cubic curve fibrations are contained in precisely a pencil of G_6 -invariant pencils of cubic hypersurfaces.*

Proof. The space of G_6 -invariant pencils of cubics is a \mathbf{P}^3 , and the G_6 -invariant scrolls singular along a triple of lines are defined by points on four lines in \mathbf{P}^3 corresponding to the four subgroups K_i . An abelian surface in the intersection of two scrolls is contained in the pencils of cubics corresponding to a line joining two of these lines. If there were more than a pencil of invariant pencils of cubics through the surface, then it would be contained in four scrolls. This is impossible (see Proposition 7.10). \square

Consider the Grassmannian of lines in the space \mathbf{P}^3 of G_6 -invariant pencils of cubics. The four lines of pencils defining G_6 -invariant scrolls are pairwise disjoint. The lines corresponding to abelian surfaces $A \in \mathcal{A}_{ij}$ define a one to one correspondence between two skew lines, so they form a conic section in the Grassmannian. Summing up we have 6 disjoint conic sections in the Grassmannian parametrizing \mathcal{A} .

We can now sum up our discussion as follows:

Proposition 7.14. *The variety \mathcal{A} consists of 6 irreducible components \mathcal{A}_{ij} indexed by the pairs $\{i, j\} \subset \{1, 2, 3, 4\}$. The incidence variety $\mathcal{I}(A, X)$ consists of 12 components \mathcal{I}_{ij}^k indexed by pairs $\{i, j\}$ and an element $k \in \{i, j\}$. For every component \mathcal{I}_{ij}^k there is a diagram*

$$\begin{array}{ccc} X^0(6) & \xrightarrow{\varphi} \mathcal{I}_{ij}^k & \xrightarrow{q} \mathcal{V}^k \\ & p \downarrow & \\ & \mathcal{A}_{ij} & \end{array}$$

where φ is $3 : 1$ and p and q are $1 : 1$.

Proof. The map φ is the map which associates to each pair $(E, (s_6, t_6))$ the abelian surface $A = E \times F$, the level 6 structure (w_1, w_2) and the scroll X which is the \mathbf{P}^2 -

scroll attached to A which is K_k -invariant. This map factors through \mathcal{A}_{ij} (giving the map ψ_{ij} of Lemma 7.9) and in particular the projection p has an inverse.

The map q is 1 : 1 by Corollary 7.12. \square

Remark 7.15. The components \mathcal{A}_{ij} , \mathcal{I}_{ij}^k and \mathcal{V}^k are all rational. We have already observed this for the varieties \mathcal{V}^k which are isomorphic to an open set of a pencil of cubics. Since the maps p and q are birational, this is also true for the other varieties.

For our conclusion on 3-folds $Y = X_E \cup X_F$ we study a G_6 -invariant rational cubic scroll. It plays a crucial role when we later bilink Y to a 3-fold of degree 7 (cf. Propositions 4.7 and 5.2).

The subgroups $H(K_i)$ have a nontrivial intersection, namely the subgroup $\langle \tau^3, \sigma^3 \rangle$. This subgroup therefore fixes all 4 triples of lines. We shall find small G_6 -orbits of planes intersecting all these lines. Consider the subgroup

$$G_3 = \langle \sigma^2, \tau^2, \iota \rangle \subset G_6.$$

We look for G_3 -invariant planes.

Now the action of τ^2 is defined by

$$\tau^2 = \text{diag}(1, \eta, \eta^2, 1, \eta, \eta^2),$$

where $\eta = e^{2\pi i/3}$ while σ^2 sends $x_i \mapsto x_{i+2}$ and ι sends $x_i \mapsto x_{-i}$. This implies easily that the 4 planes in the G_6 -orbit of any such plane must be of the form

$$\begin{array}{llllll} \alpha x_0 + \beta x_3 & = & \alpha x_2 + \beta x_5 & = & \alpha x_4 + \beta x_1 & = & 0 & P_0 \\ \alpha x_0 - \beta x_3 & = & \alpha x_2 - \beta x_5 & = & \alpha x_4 - \beta x_1 & = & 0 & P_1 \\ \beta x_0 + \alpha x_3 & = & \beta x_2 + \alpha x_5 & = & \beta x_4 + \alpha x_1 & = & 0 & Q_0 \\ \beta x_0 - \alpha x_3 & = & \beta x_2 - \alpha x_5 & = & \beta x_4 - \alpha x_1 & = & 0 & Q_1. \end{array}$$

In particular we notice that there is a 1-parameter family of such planes. The union of these planes forms a rational cubic scroll R : In fact the union of the planes is defined by

$$\text{rank} \begin{pmatrix} x_0 & x_2 & x_4 \\ x_3 & x_5 & x_1 \end{pmatrix} \leq 1,$$

so the scroll R is the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^2$ in \mathbf{P}^5 .

Remark 7.16. The orbits of planes in R do not all have length 4, in fact

$$P_0 \cap P_1 = \emptyset, \quad Q_0 \cap Q_1 = \emptyset$$

$$\text{and } P_0 \cap Q_0 = P_0 \cap Q_1 = P_1 \cap Q_0 = P_1 \cap Q_1 = \emptyset \Leftrightarrow \alpha^2 \neq \pm \beta^2.$$

Hence the orbit is 4 disjoint planes unless $\alpha^2 = \pm \beta^2$. In case $\alpha^2 = -\beta^2$ let $\alpha = 1$. Then $\beta = \pm i$ and $P_0 = Q_1, P_1 = Q_0$. While in case $\alpha^2 = \beta^2$, let $\alpha = 1$. Then $\beta = \pm 1$ and $P_0 = Q_0, P_1 = Q_1$. Thus the 4 planes are disjoint unless 2 planes

come together, which happens for

$$(\alpha, \beta) = (1, 0), (0, 1), (1, 1), (1, -1), (1, i), (1, -i), (1, 1), (1, -1),$$

altogether 4 pairs of planes.

Remark 7.17. The cubic scroll R contains all four sets of G_6 -invariant triples of lines. These lines are all transverse to the planes of R . Thus the six singular lines of two scrolls X_E and X_F which intersect along an abelian surface $E \times F$ are all contained in a rational cubic scroll.

In Lemma 8.1 we shall show that the union $Y = X_E \cup X_F$ is contained in 6 quintics. These are all singular along two triples of lines in the scroll R . We analyze these quintics more closely. With bihomogeneous coordinates s, t and y_0, y_1, y_2 on $R \cong \mathbf{P}^1 \times \mathbf{P}^2$, the restriction of quintics singular along the two triples of lines generated by τ and σ have the form

$$\begin{aligned} & y_0^2 y_1^3 - y_0^3 y_1 y_2 - y_0 y_1^2 y_2^2 + y_0^2 y_2^3 \\ & y_0^3 y_1^2 - y_0 y_1^3 y_2 - y_0^2 y_1 y_2^2 + y_1^2 y_2^3 \\ & y_0^2 y_1^2 y_2 - y_0^3 y_2^2 - y_1^3 y_2^2 + y_0 y_1 y_2^3, \end{aligned}$$

multiplied by any quintic in s, t . The 6 quintics in the ideal of two scrolls are determined by a pencil of quintics in s, t . This pencil is G_3 -invariant, so it is an element in the net

$$\langle s^5, t^5 \rangle \oplus \langle s^4 t, s t^4 \rangle \oplus \langle s^3 t^2, s^2 t^3 \rangle$$

of pencils. The basepoints of this pencil define precisely the planes of R common to all quintics through W . If the pencil has no basepoints, then the two scrolls would intersect R in only the singular lines. We shall see in Proposition 7.18 that this is not the case. On the other hand every point of intersection of the two scrolls with R outside the 6 singular lines lies in a plane which must be defined by a basepoint of the pencil, so there are at most 4 planes with such an intersection. These planes clearly form orbits under G_6 so there are 2 or 4 planes as explained above.

We carry this analysis a bit further in order to show that the union of two scrolls is bilinked to a Del Pezzo 3-fold W_7 . First we consider the intersection of the cubic scroll R and an elliptic scroll X . We use the fact that X is G_6 -invariant and that each plane in R is invariant under the subgroup $G_3 \subset G_6$ generated by $\langle \sigma^2, \tau^2 \rangle$. Thus any point in a plane has an orbit by this subgroup of order divisible by 3, and any invariant curve has degree divisible by 3.

Proposition 7.18. *The intersection $X \cap R$ is a curve of degree 18. It has the following decomposition into irreducible components:*

$$C = 2L_1 + 2L_2 + 2L_3 + l_1 + \dots + l_{12}$$

where L_i are the singular lines of X and as such meet every plane in R . The lines l_i form 4 triangles in 4 planes of the \mathbf{P}^2 -bundle R .

Proof. First we prove that the intersection is a curve. If not, it contains an irreducible surface, call it T . If T contains a plane, this is common to R and X . But no plane in X is stabilized by G_3 , in fact X intersects only one of the 4 triples of lines, so this is impossible. Thus T intersects each plane in R in some curve. This curve has degree 3 since X is contained in two cubics and the curve is invariant under G_3 .

As noted above, the three singular lines of the scroll X all lie in R , and are transverse to the planes in R . Since every cubic through X is singular along the three lines, the intersection of a cubic with R is a triangle in each plane. In fact T in $R = \mathbf{P}^2 \times \mathbf{P}^1$ must equal $T = T_0 \times \mathbf{P}^1$, where T_0 is this triangle, i.e. T is the union of three quadric surfaces. But X does not contain any quadric surface, so the intersection $X \cap R$ cannot contain a surface. It is therefore a curve, call it Γ . Since the intersection is proper, this curve has, by Bezout, degree 18.

We have seen already that the curve Γ contains the three singular lines. In fact on \tilde{X} , the normalization of X , the preimage $\tilde{\Gamma}$ of Γ contains the curves F_i which lie 2 : 1 over the singular lines L_i . Thus on \tilde{X} we have

$$\tilde{\Gamma} = F_1 + F_2 + F_3 + ah^2 + bhf,$$

in notation as in the proof of Proposition 4.10, with $6a + b = 12$. Since any plane which intersects R properly intersects in a scheme of length 3, the general plane of X must intersect R in three points, the points of intersection between the plane and the lines L_i . Therefore $a = 0$, and $\tilde{\Gamma} = F_1 + F_2 + F_3 + l_1 + \dots + l_{12}$ where the l_i are lines in the planes of \tilde{X} .

Finally if a plane in R intersects X along a curve, this curve has degree divisible by three, in fact equal three, since X lies in two cubics. Therefore the twelve lines l_i form triangles in four planes of R . \square

8. Conclusion

We shall conclude by showing that the union of two scrolls X_E and X_F which intersect along an abelian surface $E \times F$ is bilinked to a Del Pezzo W_7 .

Let $Y = X_E \cup X_F$.

Lemma 8.1. $h^0(\mathcal{I}_Y(5)) = 6$ and Y lies on irreducible quintic hypersurfaces.

Proof. Consider the exact sequence

$$0 \rightarrow \mathcal{I}_Y(5) \rightarrow \mathcal{I}_{X_E}(5) \oplus \mathcal{I}_{X_F}(5) \rightarrow \mathcal{I}_{E \times F}(5) \rightarrow 0.$$

First, the intersection $E \times F \cap \mathbf{P}^3$ of $E \times F$ with a general \mathbf{P}^3 is at least 5-regular in the sense of of Castelnuovo-Mumford, so the same is true for $E \times F$, i.e. $h^1(\mathcal{I}_{E \times F}(k)) = 0$, when $k \geq 4$. In particular, $h^0(\mathcal{I}_{E \times F}(5)) = 102$. Similarly, the intersection $Y \cap \mathbf{P}^2$ of Y with a general plane is 5-regular in the sense of

Castelnuovo-Mumford so $h^1(\mathcal{I}_Y(k)) = 0$ for $k \geq 4$. Furthermore $h^0(\mathcal{I}_{X_E}(5)) = h^0(\mathcal{I}_{X_F}(5)) = 54$ by Corollary 6.4. Therefore $h^0(\mathcal{I}_Y(5)) = 6$. If every quintic in the ideal of Y is reducible, then they all have a fixed quartic hypersurface as a component. But Y is G_6 -invariant, and there are no G_6 -invariant quartic hypersurfaces, so the lemma follows. \square

Now, according to Proposition 7.18 the two elliptic scrolls X_E and X_F intersect the rational scroll R in the six singular lines and in four triangles each. Since $X_E \cap X_F$ is a surface which does not meet the six singular lines, the two sets of four triangles must lie pairwise in the same four planes of R . Thus the six quintics through the two scrolls $X_E \cup X_F$ contain these four planes. In particular, if Z is linked $(5, 5)$ to the union $X_E \cup X_F$, then Z has degree 13 and contains four planes of R . Furthermore since each quintic intersects the planes in R in curves singular in the six points of intersection with the singular lines, Z will intersect each plane in one point in addition to the six singular points. The general quintic in the pencil intersects a plane in an irreducible curve, so, by Bezout, there are no quartic curves singular in the six singular points. Therefore any quartic hypersurface containing Z must also contain R .

Now, the partial normalization Y' of $Y = X_E \cup X_F$ along the 6 singular lines is Calabi-Yau (cf. Proposition 3.5). In particular the dualizing sheaf $\omega_{Y'}$ is trivial and has one global section. It follows from the next lemma that the dualizing sheaf ω_Y also has a section.

Let $f : Y' \rightarrow Y$ be a finite morphism of projective schemes and denote by ω_Y the dualizing sheaf of Y . Then $f^!\omega_Y$ is a dualizing sheaf of Y' (See [Ha, Ex III, 7.2] and for the definition of $f^!\omega_Y$ see [Ha, Ex III, 6.10]). Hence we can put $\omega_{Y'} = f^!\omega_Y$.

Lemma 8.2. *If $H^0(Y', \omega_{Y'}) \neq 0$, then also $H^0(Y, \omega_Y) \neq 0$.*

Proof. Using [Ha, Ex III, 6.10(b)] we have that

$$\begin{aligned} H^0(Y', \omega_{Y'}) &= \text{Hom}_{\mathcal{O}_{Y'}}(\mathcal{O}_{Y'}, \omega_{Y'}) = \text{Hom}_{\mathcal{O}_{Y'}}(\mathcal{O}_{Y'}, f^!\omega_Y) \\ &= H^0(\mathcal{H}om_{\mathcal{O}_{Y'}}(\mathcal{O}_{Y'}, f^!\omega_Y)) \\ &= H^0(f_*\mathcal{H}om_{\mathcal{O}_{Y'}}(\mathcal{O}_{Y'}, f^!\omega_Y)) \\ &= H^0(\mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_{Y'}, \omega_Y)) \quad ([Ha., Ex.III.6.10(b)]) \\ &= \text{Hom}_{\mathcal{O}_Y}(f_*\mathcal{O}_{Y'}, \omega_Y). \end{aligned}$$

Hence a section s of $\omega_{Y'}$ gives rise to a morphism $\varphi_s : f_*\mathcal{O}_{Y'} \rightarrow \omega_Y$. Combining this with the natural morphism $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_{Y'}$, we obtain a morphism $\mathcal{O}_Y \rightarrow \omega_Y$ and hence a section of ω_Y . \square

In the cohomology of the liaison exact sequence (cf. [PS])

$$0 \rightarrow \omega_Y \rightarrow \mathcal{O}_{Y \cup Z}(4) \rightarrow \mathcal{O}_Z(4) \rightarrow 0,$$

a section of ω_Y corresponds to a section of $h^0(\mathcal{I}_Z(4))$, i.e. to a quartic hypersurface containing Z . This quartic must contain R .

Let W be linked to Z in this quartic and a general quintic through Z . Then W must intersect R in a surface, in fact in a surface linked to 4 planes in the intersection of R with a quintic hypersurface. This is clearly a surface of degree 11. Furthermore the arithmetic genus of W is 1 by linkage, so $W \cup R$ has arithmetic genus 11. From the liaison exact sequences

$$0 \rightarrow \omega_Z(1) \rightarrow \mathcal{O}_{Y \cup Z}(5) \rightarrow \mathcal{O}_Y(5) \rightarrow 0,$$

$$0 \rightarrow \omega_Z(1) \rightarrow \mathcal{O}_{W \cup Z}(4) \rightarrow \mathcal{O}_W(4) \rightarrow 0$$

we get that $h^0(\mathcal{I}_W(4)) = h^0(\mathcal{I}_Y(5)) - 1 = 5$.

Clearly all the quartics through W have to contain also R . Thus $W \cup R$ is contained in 5 quartics.

For later we need that the 5 quartics are minors of a 4×5 matrix with linear entries. If we can show this in a special case, we may conclude by semicontinuity that so is $W \cup R$ in general (cf. [Ell]).

We do this by considering, in the notation of Remark 7.8, the points $(1, -3, 0, 0)$ and $(0, 0, 0, 1)$ in the \mathbf{P}^3 of G_6 -invariant pencils of cubics. They are points on two distinct lines corresponding to two distinct subgroups $H(K_i)$. The corresponding pencils of cubics each define an elliptic scroll (in fact a reducible scroll) residual to three \mathbf{P}^3 's. The union of these scrolls lies on 6 quintics and is bilinked $(5, 5)$ and $(5, 4)$ to a 3-fold of degree 7 which lies in 5 quartic hypersurfaces. These 5 quartics define a determinantal 3-fold of degree 10. These claims are easily checked with i.e. [MAC].

Recall from Proposition 4.10 that W is a non-normal Del-Pezzo 3-fold W_7 as soon as we have checked that no three of the singular lines lie in a \mathbf{P}^3 , and that not all six lie in a rational normal quartic scroll, and finally that the common singular locus of the quartic hypersurfaces through W is precisely the 6 singular lines. While the former two requirements follows easily from our analysis of the singular lines in the previous section, the latter requirement is easily checked in the above example. Therefore we may, by Proposition 5.2, bilink W in complete intersections $(5, 4)$ and $(5, 5)$ to a non-normal Calabi-Yau 3-fold Y_t , non-normal only along 6 lines. Clearly, we may perform the bilinkage in a family, so the reducible 3-fold Y is a degeneration of Y_t . We have shown

Theorem 8.3. *The reducible 3-fold $Y = X_E \cup X_F$ is a degeneration of irreducible non-normal Calabi-Yau 3-folds of degree 12 in \mathbf{P}^5 . The general such 3-fold is singular precisely along 6 disjoint lines.*

Remark 8.4. Computing the normalizer N_3 of $G_3 \subset GL(6, \mathbf{C})$ and its representations, one may show that there is an N_3 -invariant linear complex in the Grassmannian G of lines in the space of G_6 -invariant pencils of cubics. This defines a one to one correspondence between any two of the four lines defined by the subgroups K_i . Therefore it is natural to guess that this complex defines the correspondences

\mathcal{A}_{ij} , in fact that it parametrizes the set of all G_6 -invariant abelian surfaces. This is proved by Gross and Popescu [GP2].

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